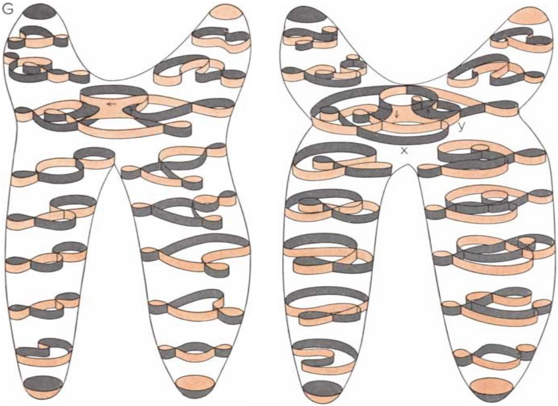
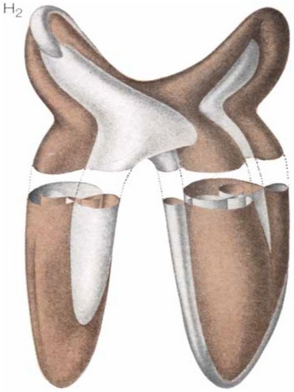
Sphere eversions & the h-principle

Lauran Toussaint - VU General Math Colloquium





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H-principle approach: look at the space of all solutions at once.

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$\Rightarrow$ **homotopy**: Allow solutions to be continuously deformed.

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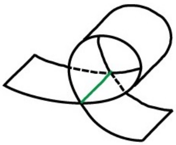

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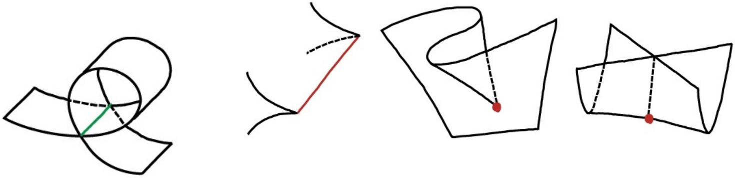

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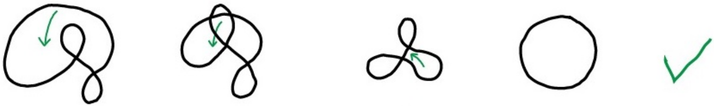

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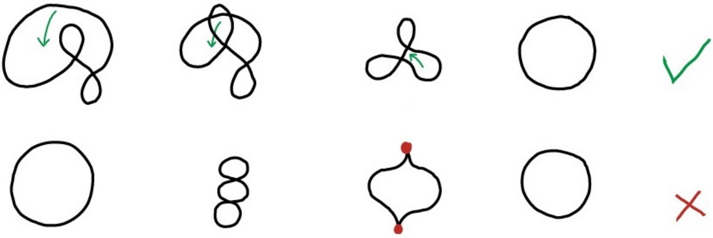

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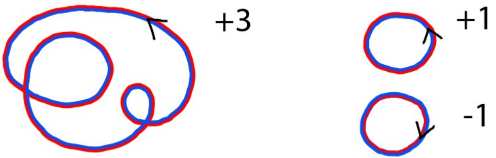

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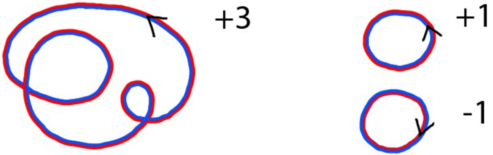

- Rotation number is invariant under regular homotopy.

Question:

Is there an *explicit* sphere eversion?

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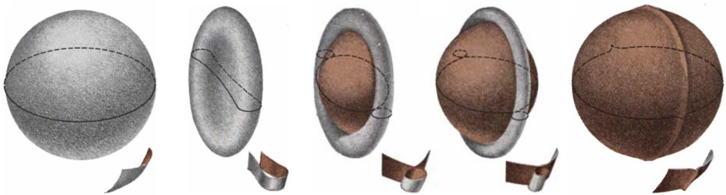
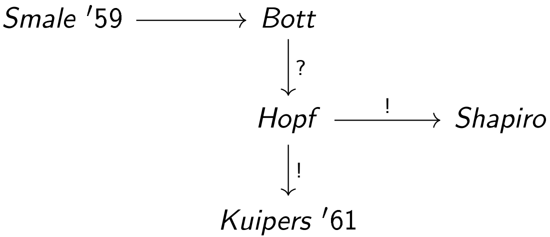
- The naive try does not work due to cusps:



Smale '59

$Smale\ '59 \longrightarrow Bott$

$$\begin{array}{ccc} \textit{Smale '59} & \longrightarrow & \textit{Bott} \\ & & \big\downarrow{\scriptstyle ?} \\ & & \textit{Hopf} \end{array}$$


Smale '59
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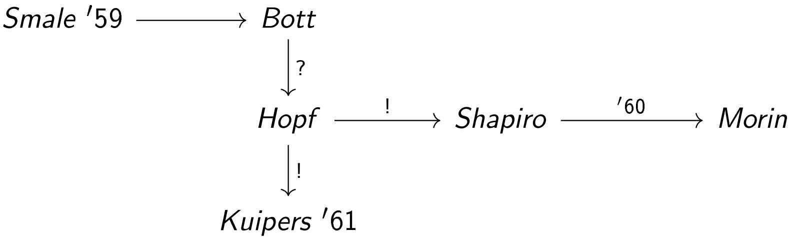
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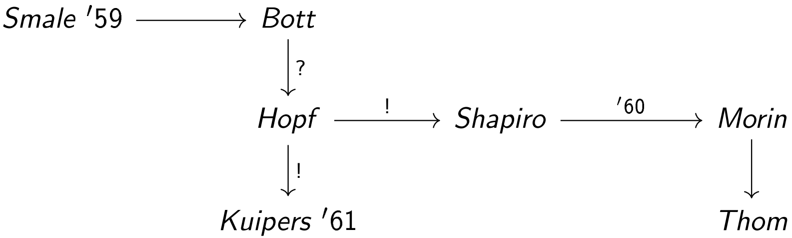
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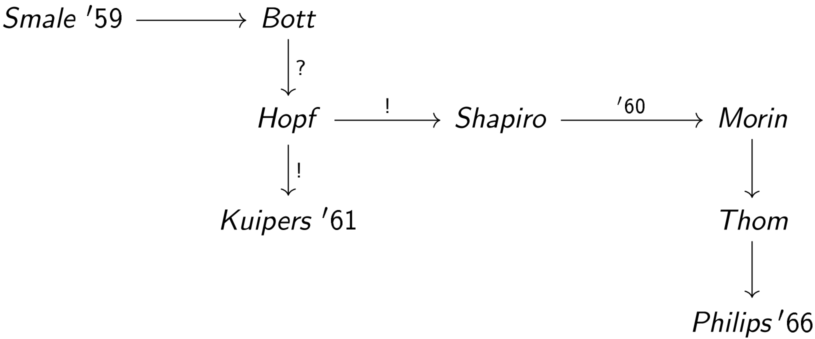
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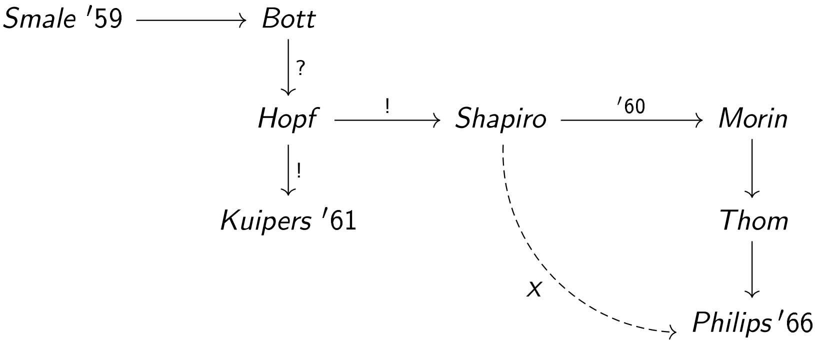
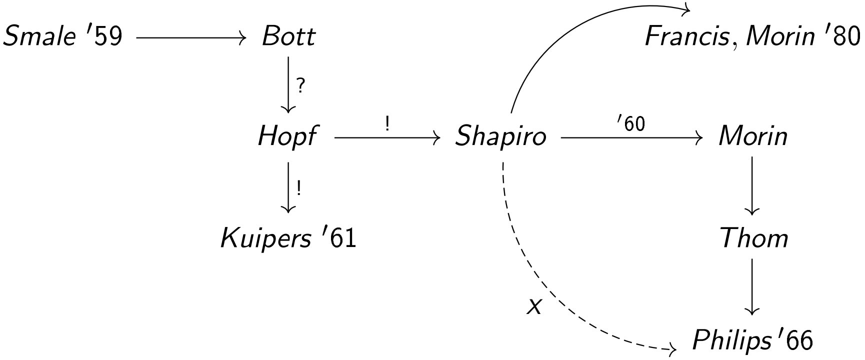


In fact, Smale's paper contains no pictures at all. The intricacy of the pictures, which were in a sense implicit in Smale's abstract and analytical mathematics, is amazing. Perhaps even more amazing is the ability of mathematicians to convey these ideas to one another without relying on pictures. This ability is strikingly brought out by the history of Shapiro's description of how to turn a sphere inside out. I learned of its construction from the French topologist René Thom, who learned of it from his colleague Bernard Morin, who learned of it from Arnold Shapiro himself. Bernard Morin is blind.



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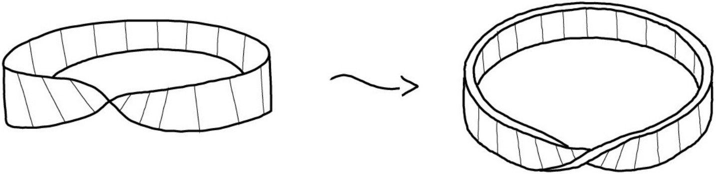

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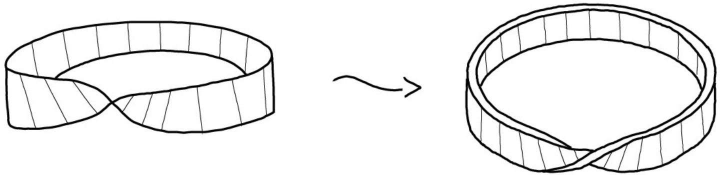

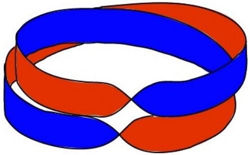

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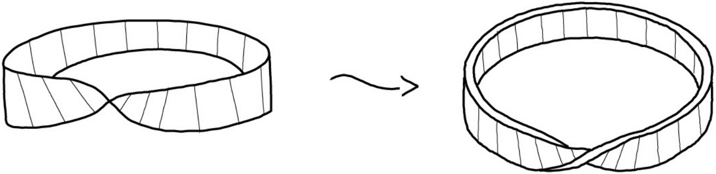
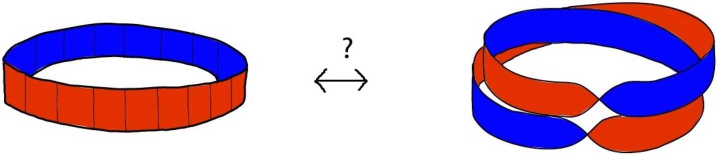


- Regular homotopic to standard cylinder?



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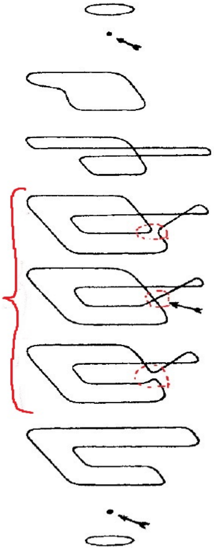

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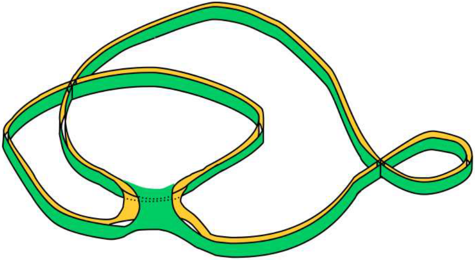

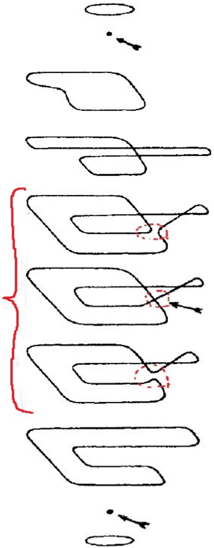

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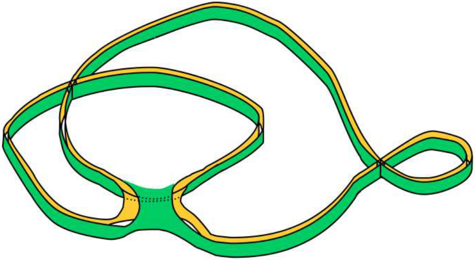
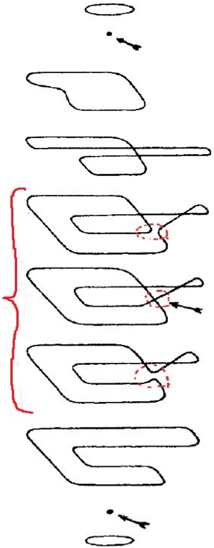
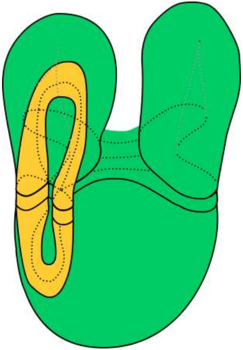
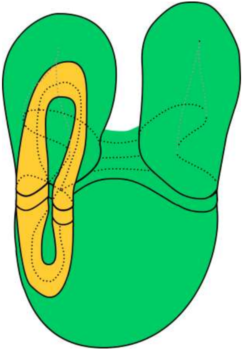


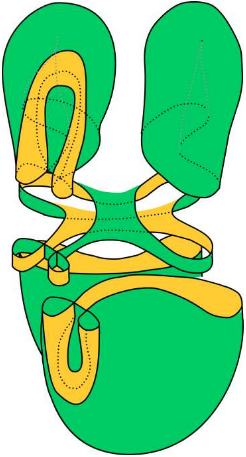
- Philips figured out how to complete the movie.

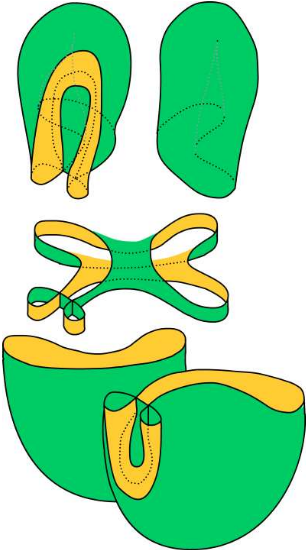


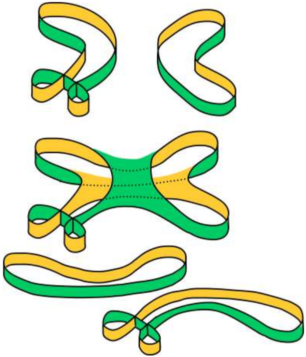
Philip's pictures:

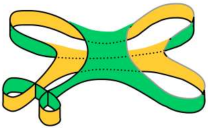


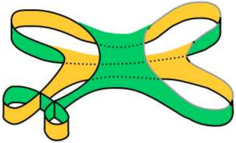


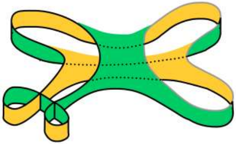


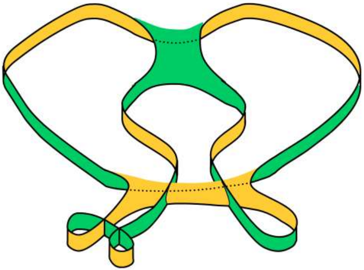


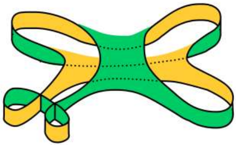


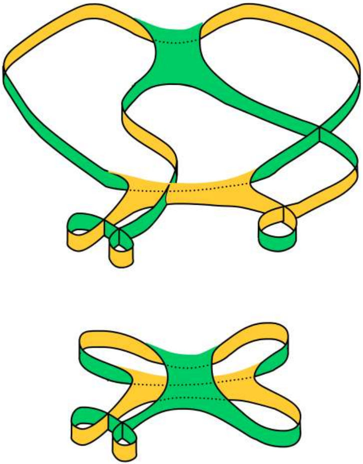


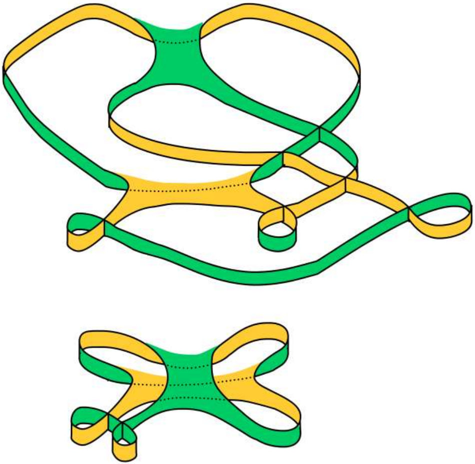


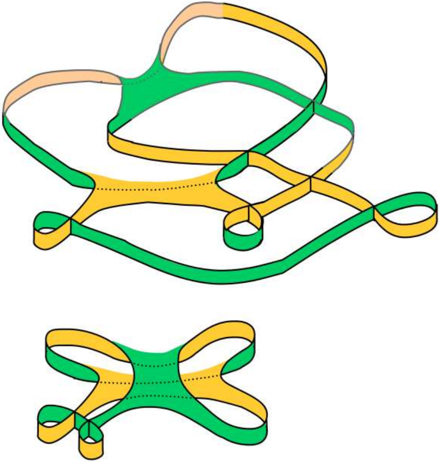


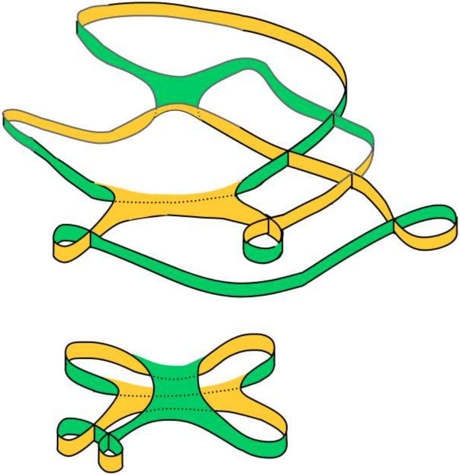


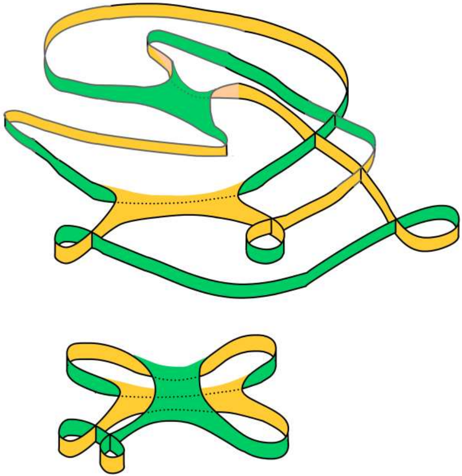


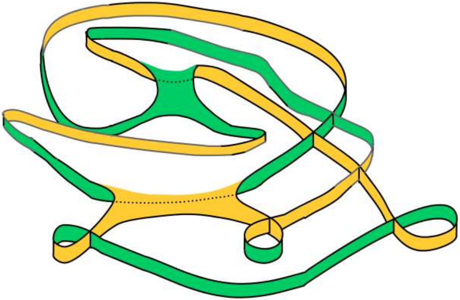


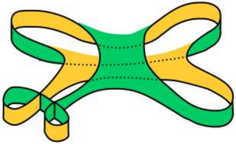


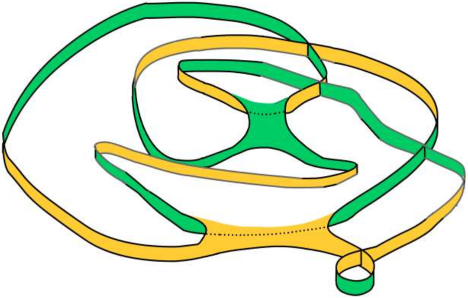


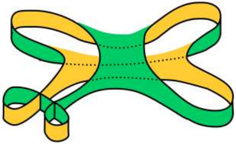


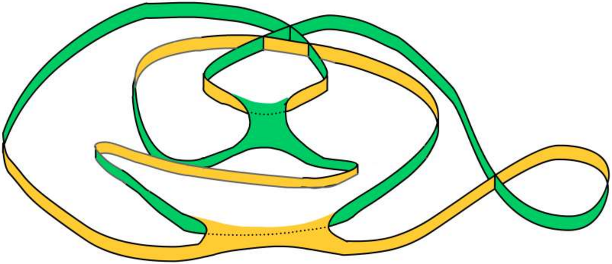


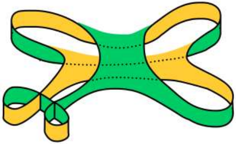


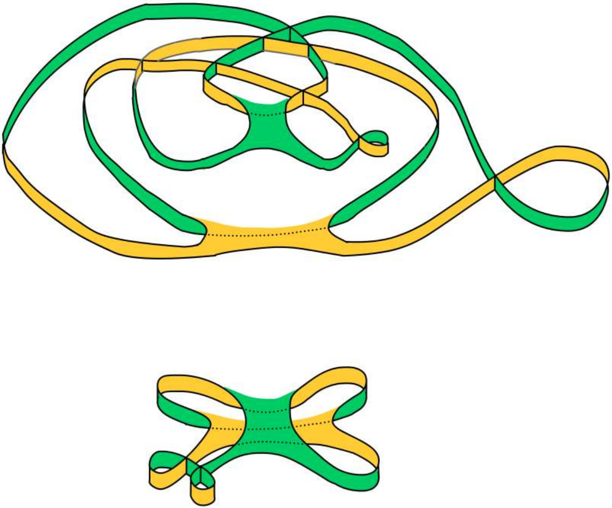


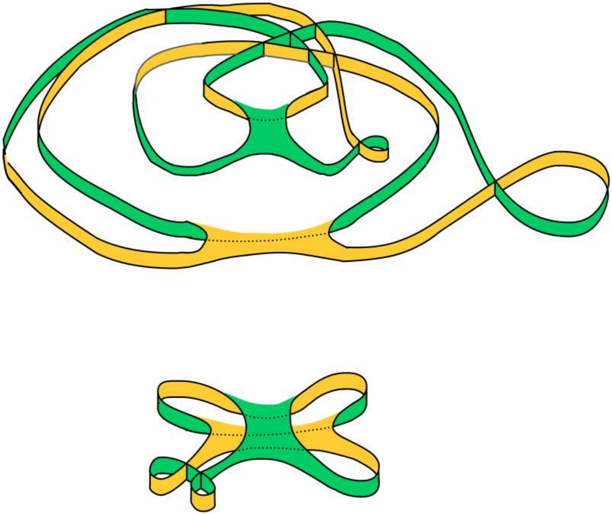


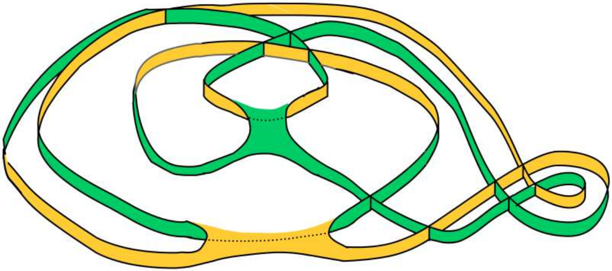


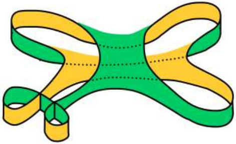


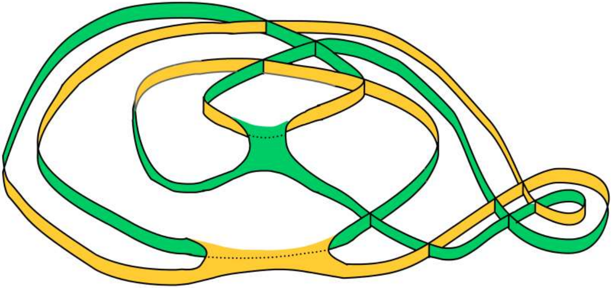


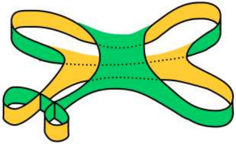


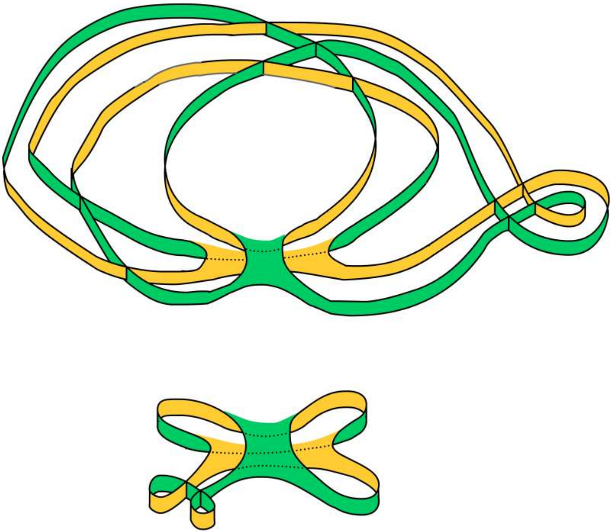


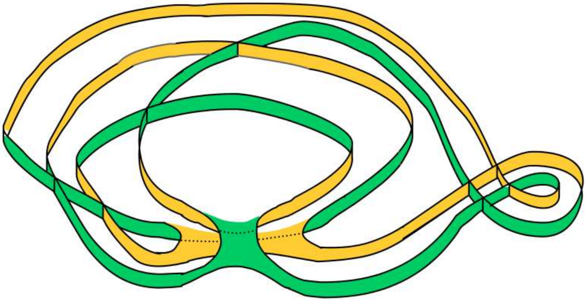


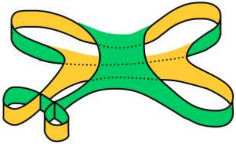


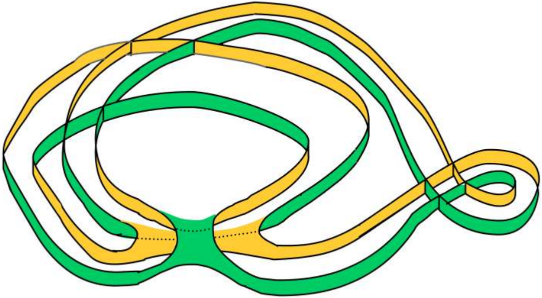


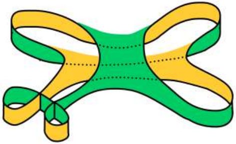


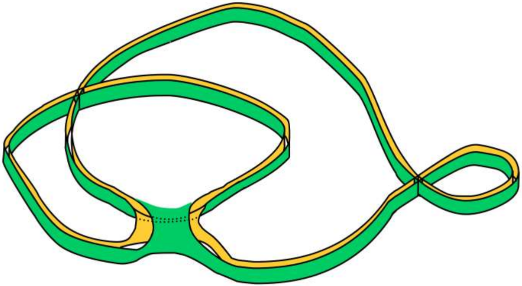


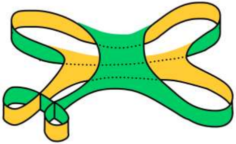


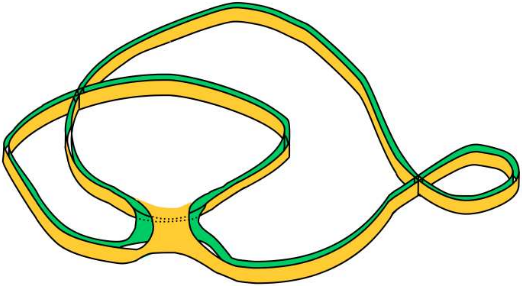


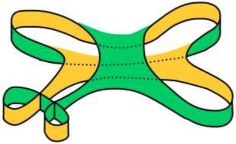


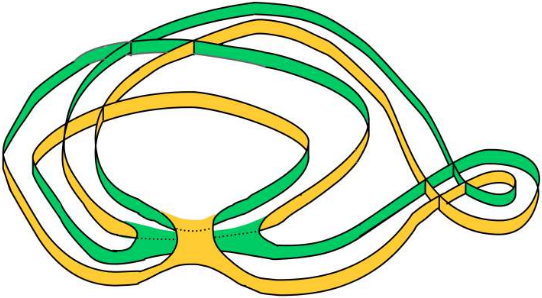


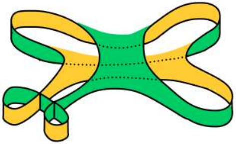


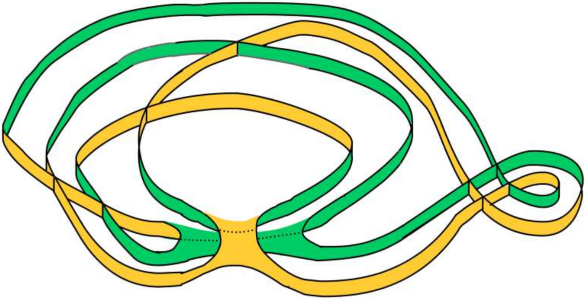


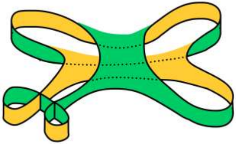


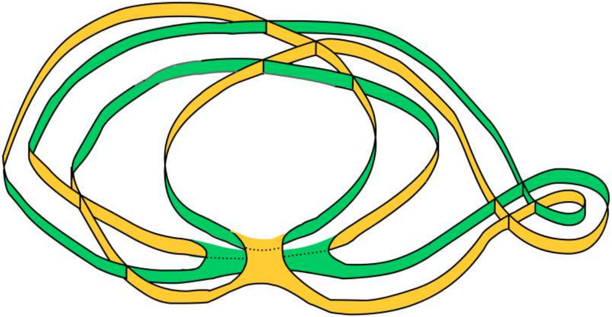


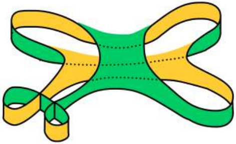


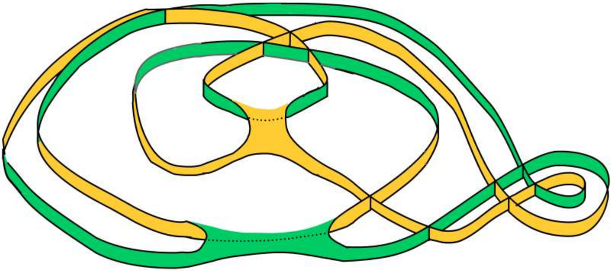


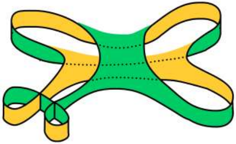


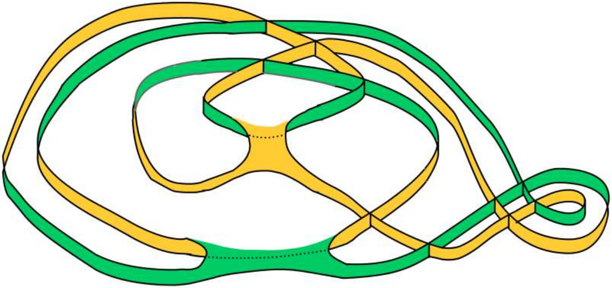


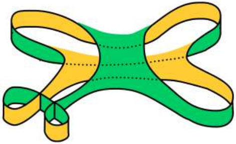


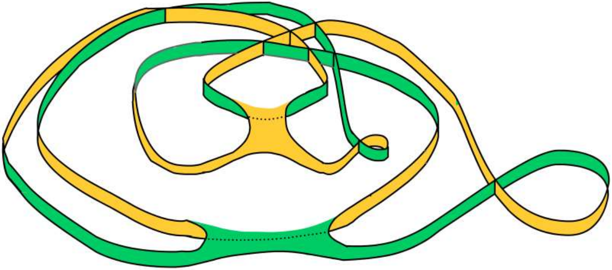


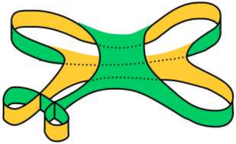


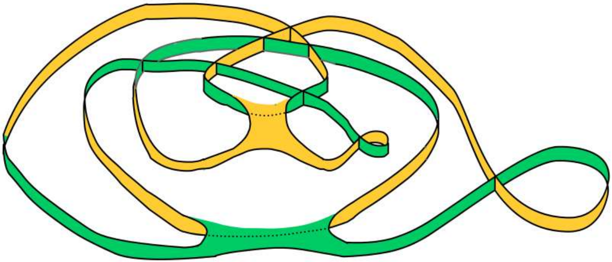


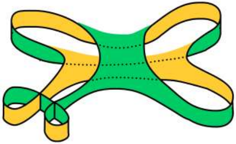


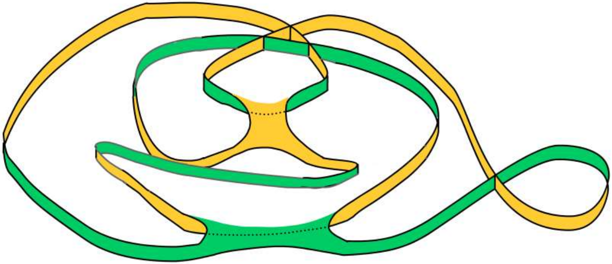


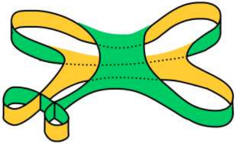


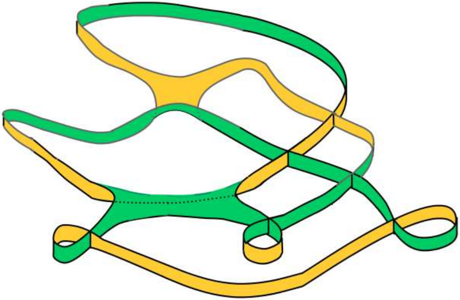


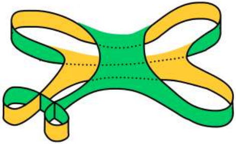


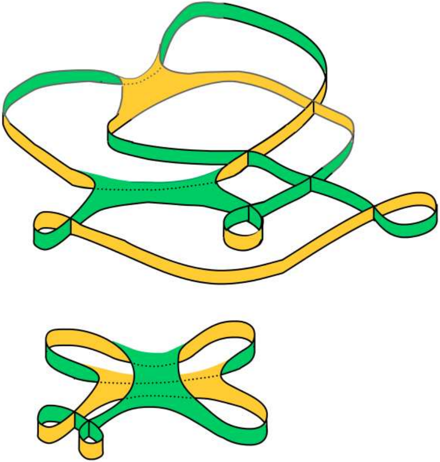


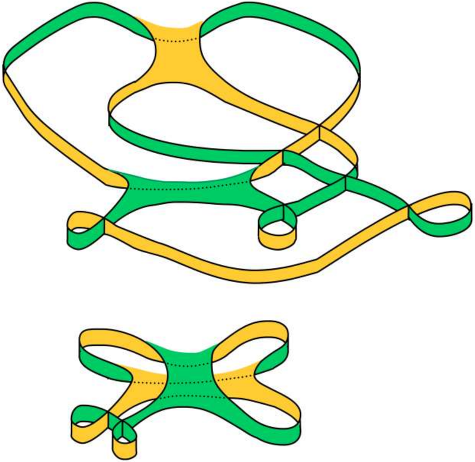


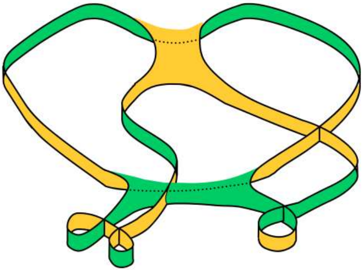


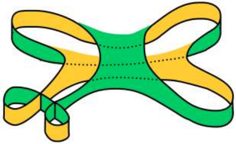


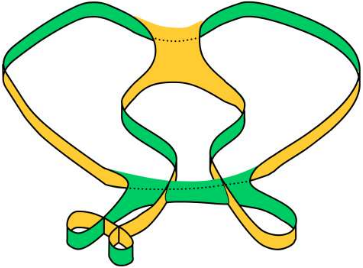


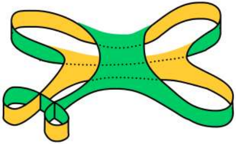


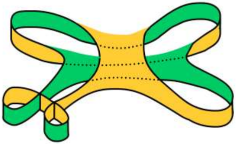


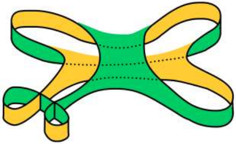




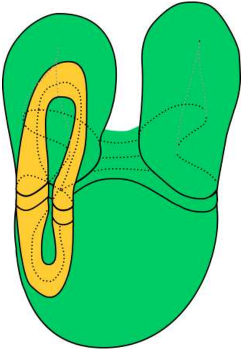


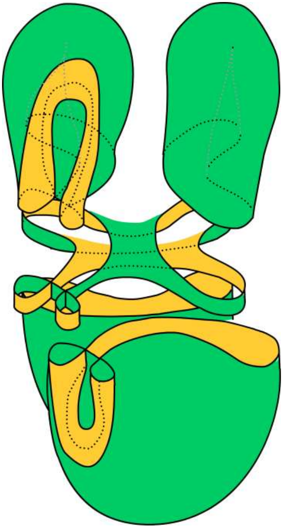


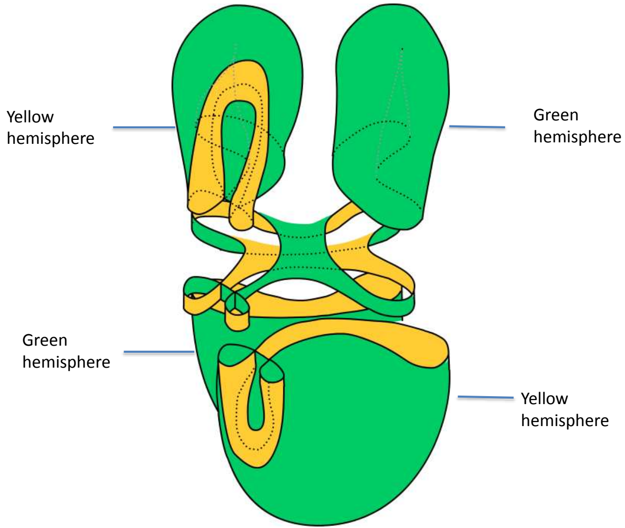


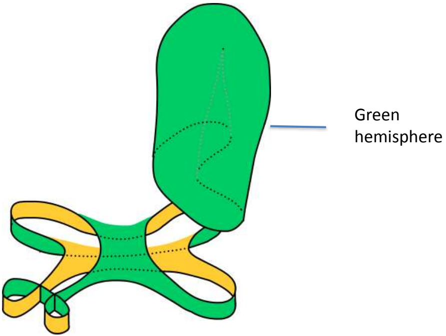


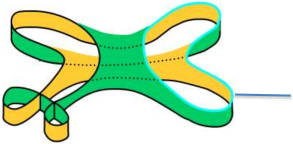
What happens to the rest of the surface?



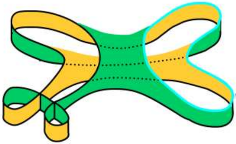



Yellow
hemisphere
Green
hemisphere
Green
hemisphere
Yellow
hemisphere


Green
hemisphere



Green
hemisphere
attached here


Green
hemisphere
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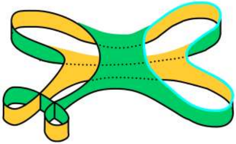
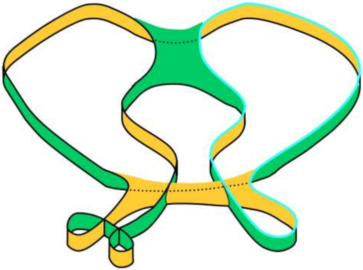
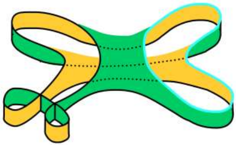
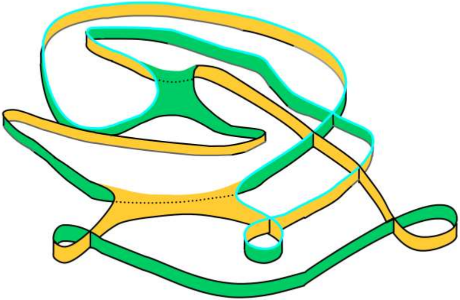
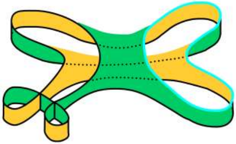
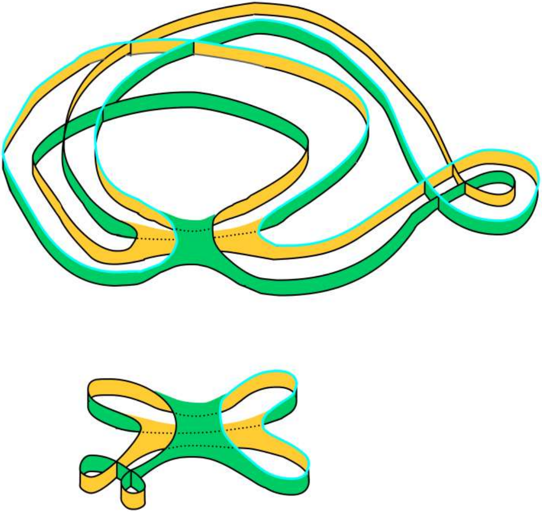


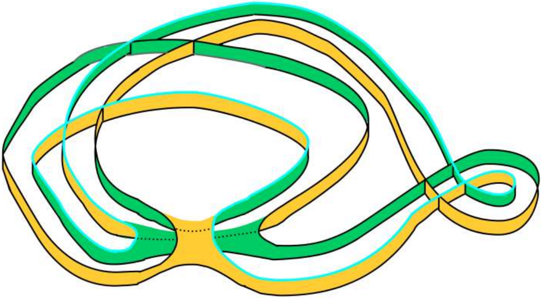


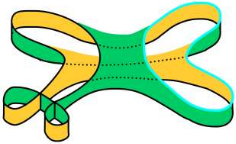


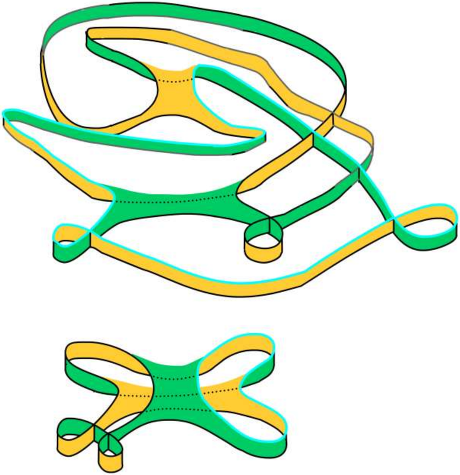


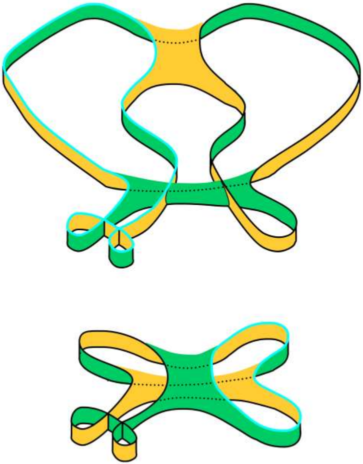




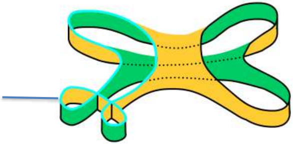


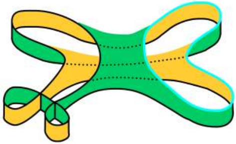


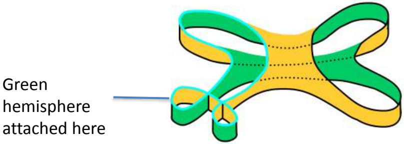




Green
hemisphere
attached here



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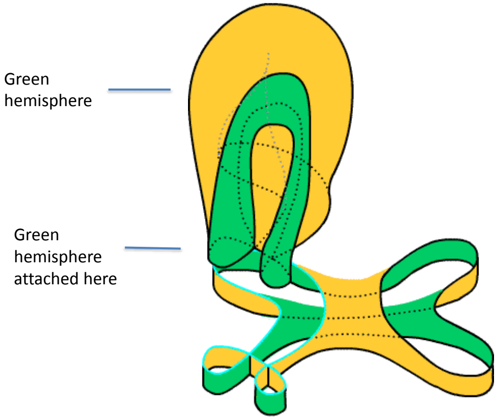
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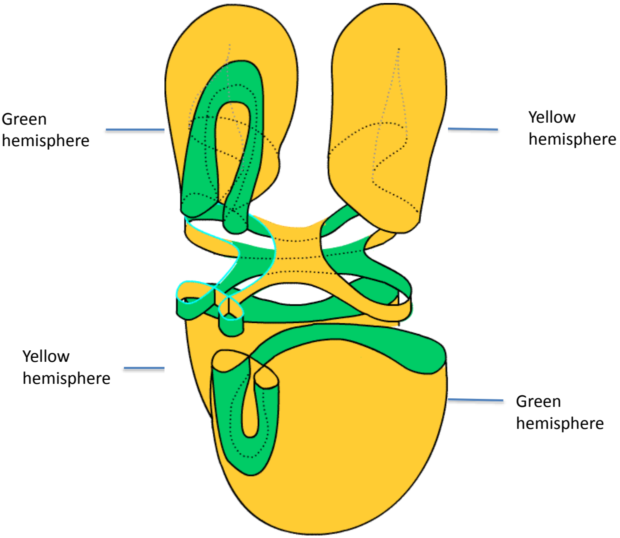
Green
hemisphere
Yellow
hemisphere
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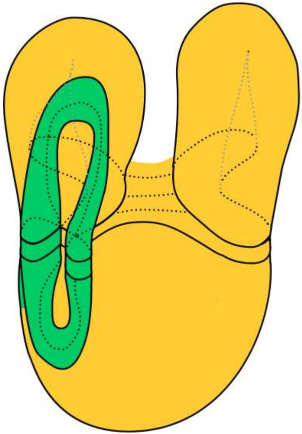

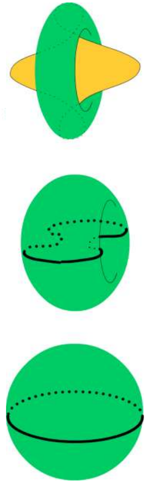

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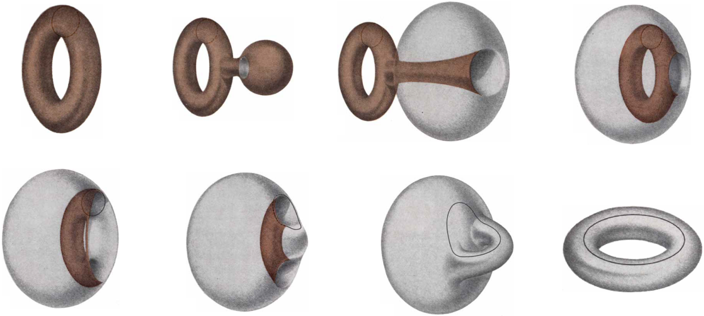
- Any (genus g) surface can be turned inside out. For example, torus:



Definition

The r-**th jet space** of functions $f : M \to \mathbb{R}$

$$\begin{array}{ccl} J^r(M,\mathbb{R}) & := & \{r\text{-th order Taylor polynomials of functions}\} \\ & = & \text{tuples } (x, y, y', y'', \dots) \\ \Big\downarrow \pi & & \qquad\qquad\qquad\qquad \Big\downarrow \\ M & & \qquad \text{basepoint } x \text{ of the Taylor polynomial} \end{array}$$

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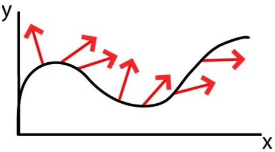

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- **Holonomic sections:** Given f we get a section $j^r f : M \to J^r(M,\mathbb{R})$
$$x \mapsto (x, f(x), f'(x), f''(x), \dots).$$

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A **differential relation** of order r is a subset $\mathcal{R} \subset J^r(M, \mathbb{R}^n)$.

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Differential equation:

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induces

$$\mathcal{R}_F := \{(x, y, z) \mid F(x, y, z) = 0\} \subset J^1(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^3$$

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$\mathcal{R}$ satisfies the **h-principle** if $\iota_{\mathcal{R}}$ is an isomorphism (up to homotopy).

- **Formal solutions:** $\mathrm{Sol}^f_{\mathcal{R}}(M) := \{\text{sections } f : M \to \mathcal{R}\}$.
  Dont always exist:
  $$f^2 + \dot{f}^2 - 1 = 0 \rightsquigarrow \mathcal{R} := \{x^2 + y^2 + 1 = 0\} \subset J^1(\mathbb{R}, \mathbb{R}),$$
  has no formal solutions.
- **Solutions:** $\mathrm{Sol}_{\mathcal{R}}(M) := \{f \in \mathrm{Sol}^f(\mathcal{R}) \mid f \text{ holonomic}\}$.

There is a natural inclusion:

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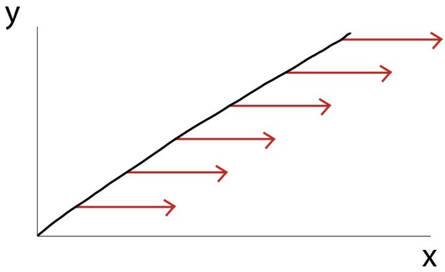
Definition

$\mathcal{R}$ satisfies the **h-principle** if $\iota_{\mathcal{R}}$ is an isomorphism (up to homotopy).

- Any formal solution is homotopic to a solution.
- Two solutions are homotopic if and only if they are homotopic as formal solutions.

Consider the following (formal) section of $J^1(\mathbb{R}, \mathbb{R})$ (we ignore $\mathcal{R}$ for now).



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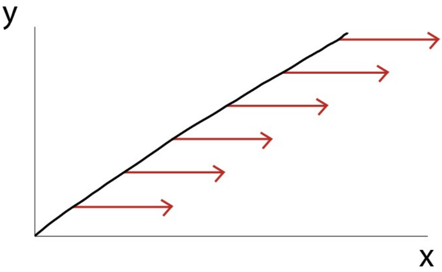

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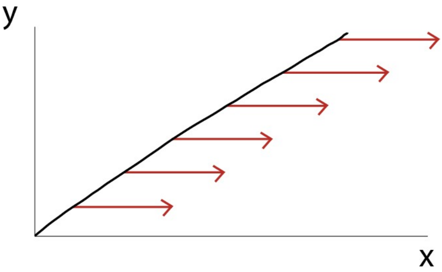


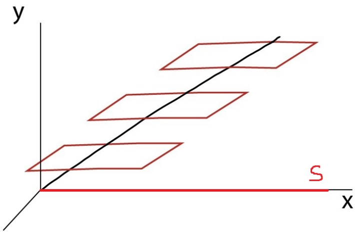
Question

Is there a holonomic section approximating it?

$\Rightarrow$ No, if $\dot{f} < \varepsilon$ then

$$f(1) - f(0) = \int_0^1 \dot{f} \mathrm{d}x < \varepsilon.$$

⇒ Add more space!



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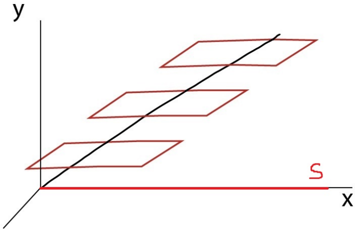
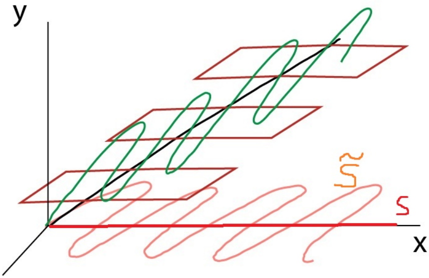


⇒ Now we can wiggle



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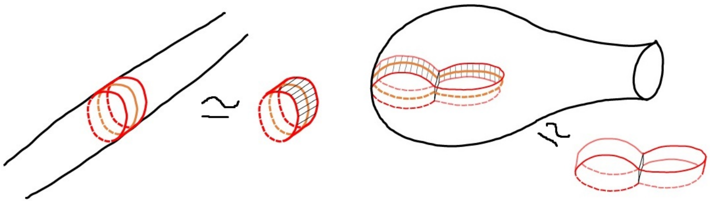
Suppose now we are given:

- $\mathcal{R} \subset J^r(M, \mathbb{R})$ open,
- $\sigma \in \mathrm{Sol}^f(\mathcal{R})$,

$\Rightarrow f \in \mathrm{Sol}(\mathcal{R})$ on a neighborhood of $\widetilde{S}$.

Moreover, if M is open (i.e. non-compact or with boundary)

- $M \simeq$ neighborhood of $S \simeq$ neighborhood of $\widetilde{S}$



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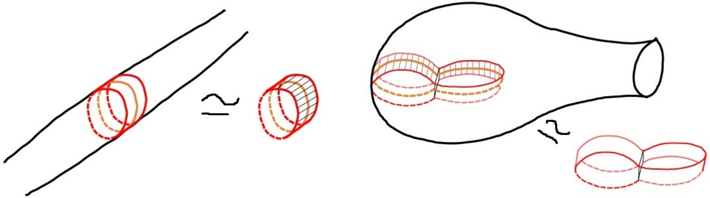

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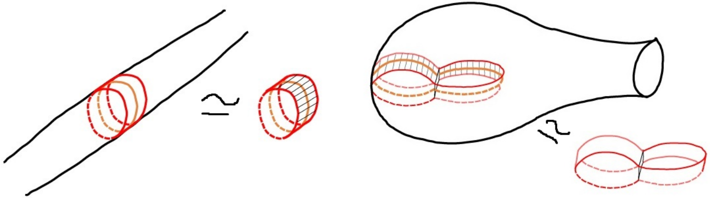

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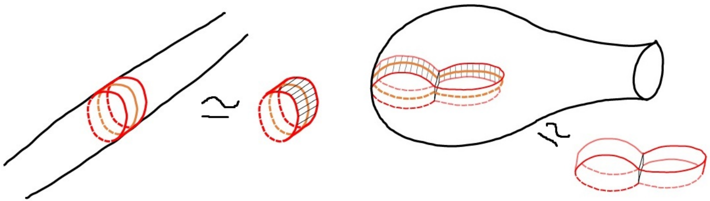

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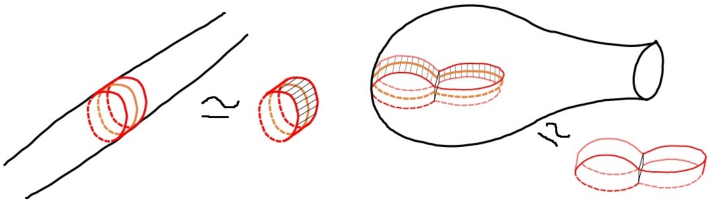

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- Microextension: $\mathbb{S}^2 \times (-\varepsilon, \varepsilon)$ is open, $\mathcal{R}_{\text{imm}}$ is open and iso-invariant.
- Fact: Any two immersions $\mathbb{S}^2 \to \mathbb{R}^3$ are formally homotopic.

  $\Rightarrow$ Smale's result.

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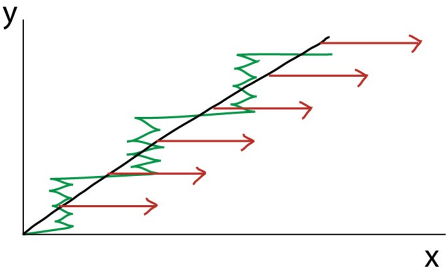

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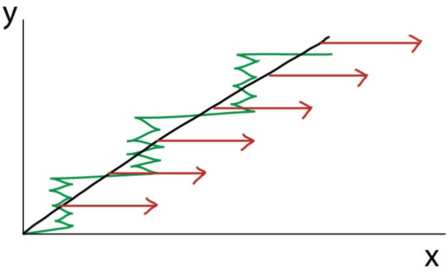

Theorem (del Pino, T.)

Any section $M \to J^r(M, \mathbb{R})$ (with M open or closed) can be approximated by a singular holonomic multi-section.

This implies the inclusion map factors:

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Theorem (Fokma, del Pino, T.)

If $\mathcal{R}$ is an open, isomorphism-invariant relation (on any M) then

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- Can also use it to simplify singularities.
- Resolving singularities much harder, sometimes possible.