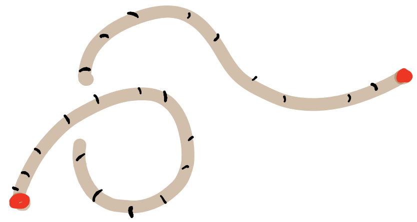
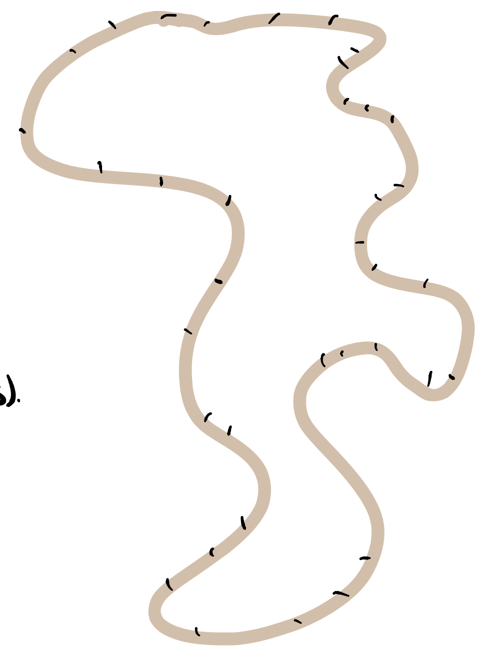
Most ropes have two ends

joint with Federica Pasquotto
(Leiden)





Programme

- My favourite theorem(s).
- Escaping a maze.
- Joey's problem
- A little bit of math: mod 2-degree theory
- A solution to Joey's problem
- Equivariant mathematics.
- Degree theory for orbifolds.

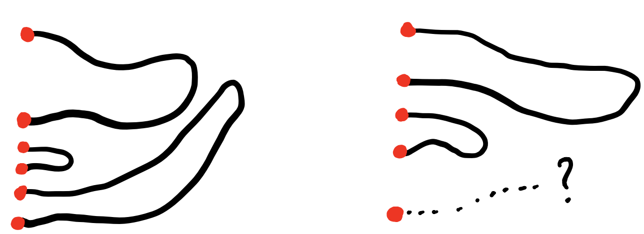
My favourite theorem:

Thm: Let M be a connected, compact manifold of dimension one. Then M is diffeomorphic to

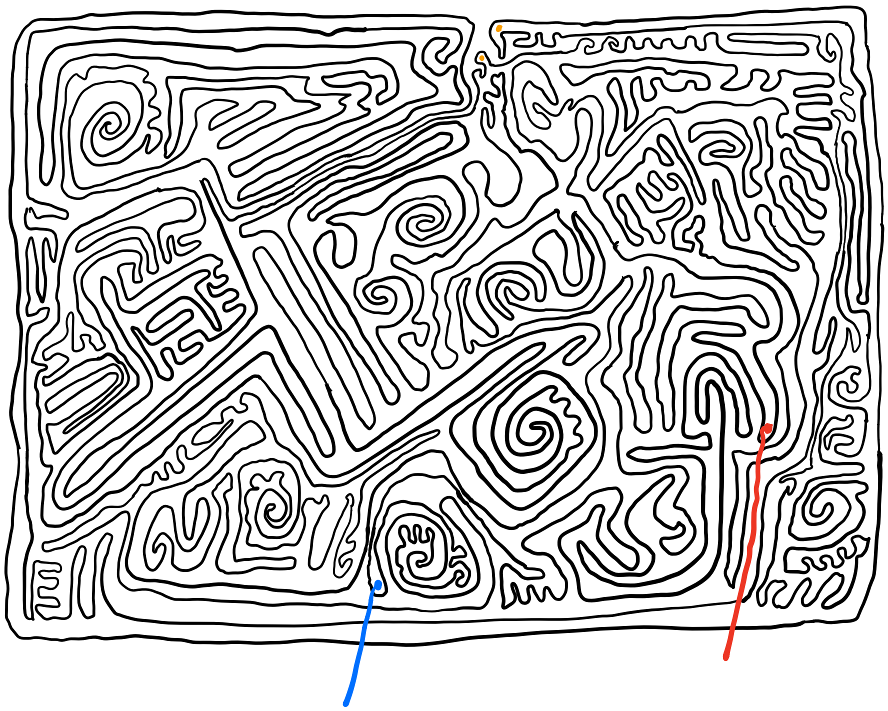
$[0,1]$ or S^1

Corollary: The number of boundary components of a one-dim compact mfd is even.

Corollary: A compact zero dimensional compact manifold is the boundary of a compact one dimensional manifold iff it has an even number of points



Can I escape this maze?

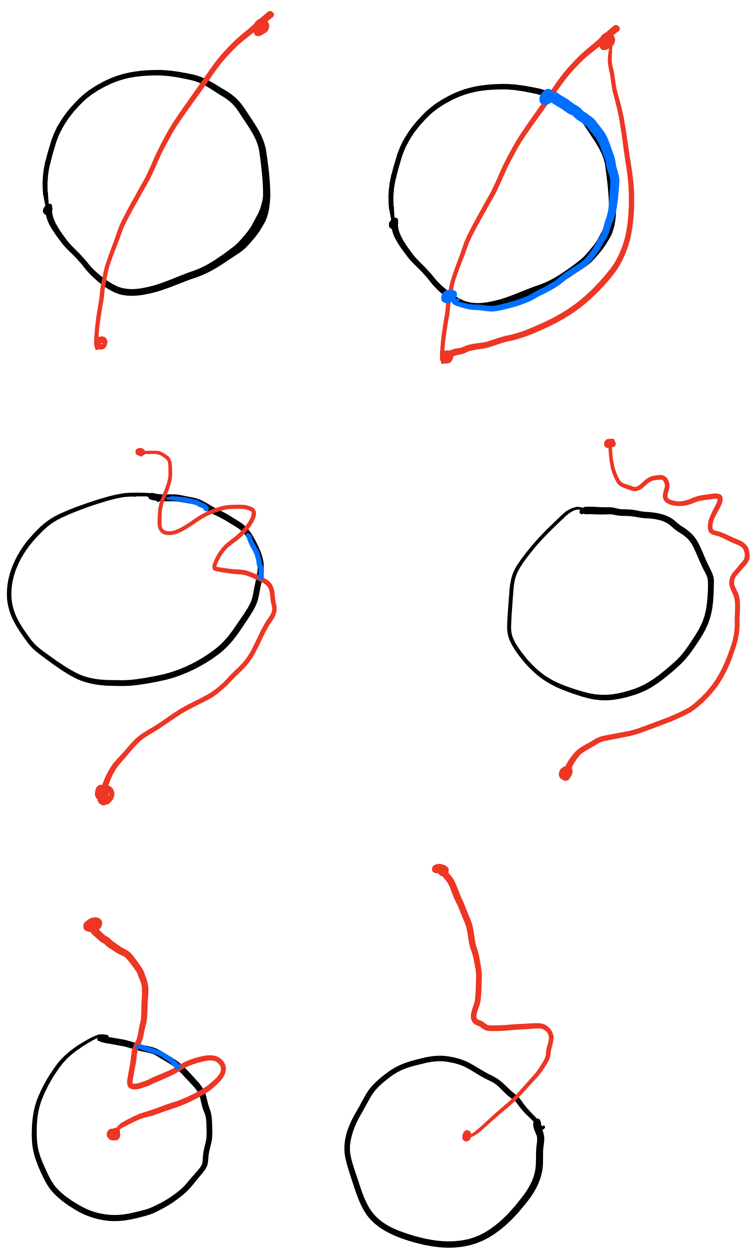


\# intersection 0 mod 2 ⇒ I can escape this maze

\# intersection 1 mod 2 ⇒ I cannot escape

Solutions

A simpler maze.



Enough with the thinking

Let's have a break.

Question: Why can Joey do this?

Towards a solution to Joey's problem

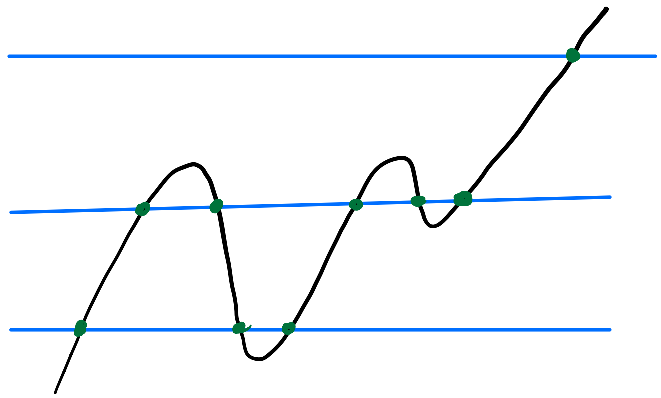
A mathy application of ropes having an even number of ends

$f: \mathbb{R} \to \mathbb{R}$

how many solutions does

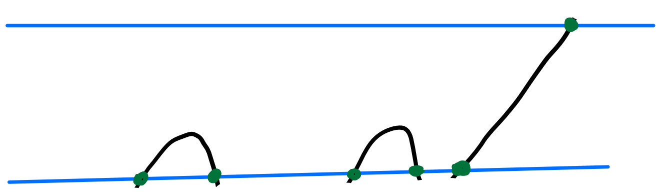
$f(x) = c$

have?



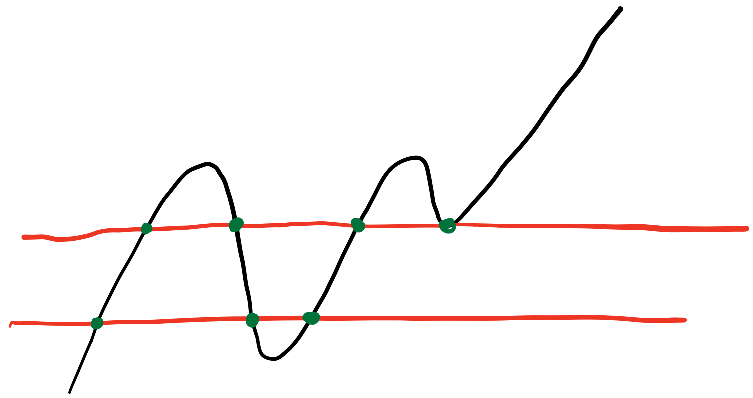
$\boxed{\textbf{Observation: } 1 \equiv 5 \equiv 3 \mod 2}$

Why is this true?

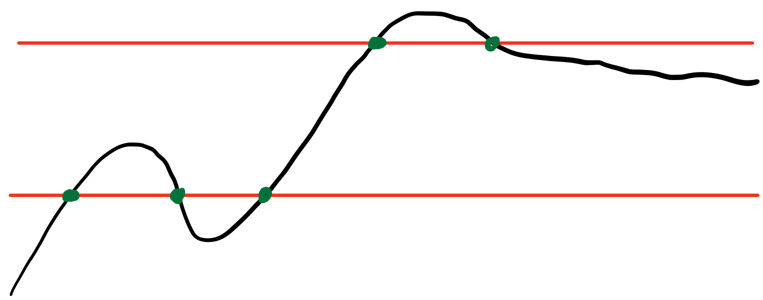


What can go wrong?

Works only for generic values (Sard's theorem)



need growth behavior at $\pm\infty$ (proper maps or closed manifolds.)

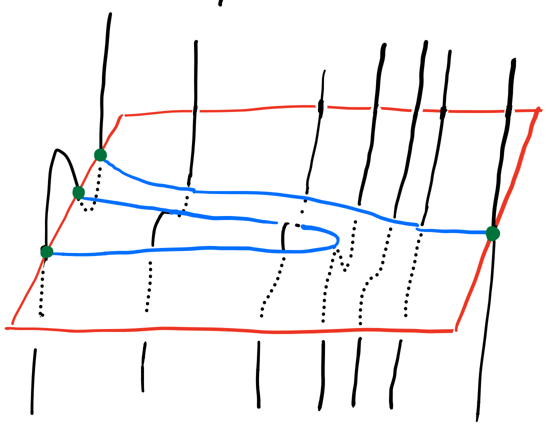


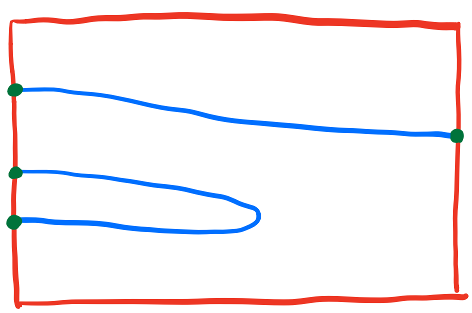
$[0,1)$

Under these conditions the count mod 2 is well-defined

This count is stable under perturbations:

$H: [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ $\quad H(t,x) = (1-t) f(x) + t g(x)$





The preimage of a regular value of H is a one-dim mfd with boundary

$$H^{-1}(y) \cap \{0\} \times \mathbb{R} = f^{-1}(y)$$

$$H^{-1}(y) \cap \{1\} \times \mathbb{R} = g^{-1}(y)$$

$$\# f^{-1}(y) + \# g^{-1}(y) = 0 \mod 2.$$

$$\# f^{-1}(y) = \# g^{-1}(y) \mod 2.$$

The mod-2 degree

Def Let $f: M^n \to N^n$ connected smooth map (proper if M & N non compact)

The mod-2 degree is

$$\deg_2(f) = \# f^{-1}(y) \mod 2.$$

for some y regular

Observations:

1) Independent of regular value y
2) Independent of homotopy i.e

$$H : [0,1] \times M \to N$$

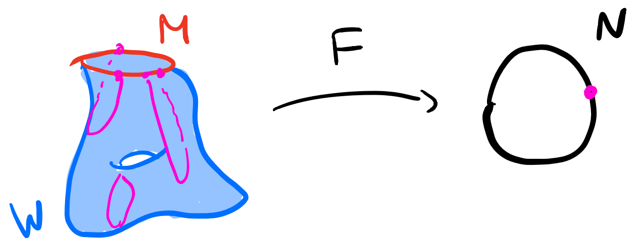
Then $\deg_2 f = \deg_2 g$ where $f(x) = H(0,x)$
$g(x) = H(1,x)$.

3) If $\deg_2 f = 1$ then f is surjective

$$\Longrightarrow$$

$f(x) = c$ has a solution for all $c \in N$!

Extensions of mappings and the degree

4) 

Suppose $M = \partial W$ and f extends over W $\quad F : W \longrightarrow N$

$$\left(F|_{\partial W} = f : M \longrightarrow N \right)$$

Then $\deg_2 f = 0$ $\boxed{\text{Why?}}$

5 For any compact manifold W with boundary ∂W

There is no map

$$r : W \longrightarrow \partial W$$

s.t. $r|_{\partial W} = id_{\partial W}$.

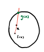
$\boxed{\text{Why?}}$

A Theorem to help Joey from friends.

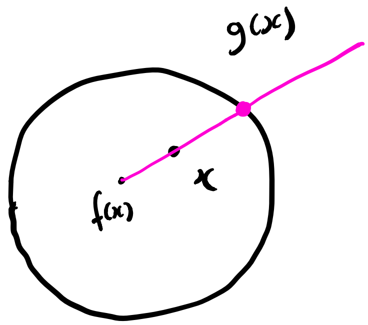
Thm (Brouwer) Every map $f: D^2 \longrightarrow D^2$ has a fixed point (i.e $\exists\, x \in D^2$ st $f(x) = x$)

(Joey can always stand on the map)

Pf: suppose $x \neq f(x)$ $\forall x$ (Joey can't stand on the map)



Construct a map $g: D^2 \longrightarrow S^1$



$x \in S^1$ $\qquad g(x) = x$ $\qquad g\big|_{\partial D^2} = id$

$\lightning$

Many many consequences: Topology (of course), Dynamical systems (Poincaré-Bendixson, ...), Game theory (Nash equilibria...), ...

A proof of the fundamental theorem of algebra.

Thm A odd degree non-constant polynomial

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0 \quad (a_i \in \mathbb{C})$$

has a root: $\exists\, z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$

Pf $p : \mathbb{C} \to \mathbb{C}$

$$H : [0,1] \times \mathbb{C} \to \mathbb{C}$$

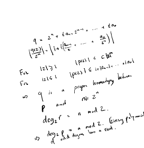
$$H(t,z) = z^n + (1-t) a_{n-1} z^{n-1} + \cdots + (1-t) a_0$$

$$H(0,z) = p(z)$$

$$H(1,z) = z^n$$

solutions to $z^n = 1$

$$\deg_2(z^n) = n \mod 2$$



(The integer valued degree)

suppose f maps between oriented manifolds.

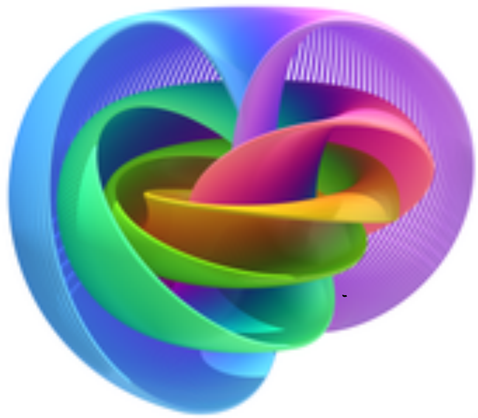
Then $f^{-1}(y)$ carries orientation.

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x$$

$$\deg f \bmod 2 = \deg_2 f.$$

This gadget can be used to prove the full FTA.

Enough with the thinking.



Let's have another break.

Equivariant topology

Common theme in mathematics.

- M parametrizes objects one is interested in
  - (Moduli space of complex structures, elliptic curves, metrics with curvature conditions, orbits in dyn system)
- Objects posses symmetry.
  (E.g. by reparametrization)

$\Rightarrow$ group G acting on M.

- True interest is in M/G
- Want to do calculus/geometry on M/G

When is this possible?

Ex <u>The Hopf fibration</u>

$z \in S^1 \subseteq \mathbb{C}$ acts on $S^3 \subseteq \mathbb{C}^2$

via $z(v, w) = (zv, zw)$

$\left(\mathbb{C}P^1 = S^3/S^1 \text{ parametrizes complex lines in } \mathbb{C}^2 \right)$

$$h : S^3 \longrightarrow S^2 \subset \mathbb{C} \times \mathbb{R}$$
$$h(v,w) = (2v\bar{w}, |v|^2 - |w|^2)$$

- Identifies S^3/S^1 with the <u>manifold</u> S^2

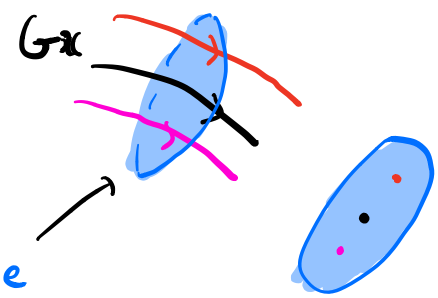
When is M/G a manifold?

- When is M/G a manifold?

M, G compact and G acts freely.

$g \cdot x \neq x \quad \forall\ x \in M \quad g \in G.$

Idea:

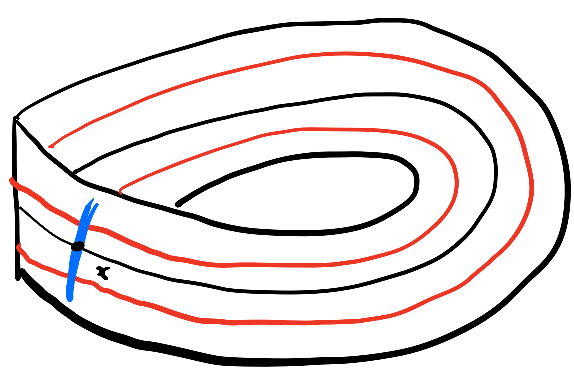


gives a chart for M/G

If $G_x = \{ g \in G \mid gx = x \}$ is trivial

every orbit close to the orbit through x intersects the orbit uniquely.

But there are situations where $G_x \neq 0$



$G_x = \mathbb{Z}_2$.

Nearby orbits can intersect the slice multiple times.

G_x acts on the slice.

Orbifolds

Charts for orbit space should be $\mathbb{R}^{d} / G_x$

d dim slice (depends on dim G_x)

$\boxed{\text{Simplifying assumption: } G_x \text{ is finite} \quad (\text{so } d \text{ is constant})}$

Def: An d-dim orbifold $\mathcal{O}$ is a space locally modelled on $\mathbb{R}^d / G_x$ $(x \in \mathcal{O})$

finite

. .

+ technical conditions

Thm: Any semi-free action of G on M defines orbifold structure on M/G.

finite stabilizers

Ex **Weighted Hopf action**

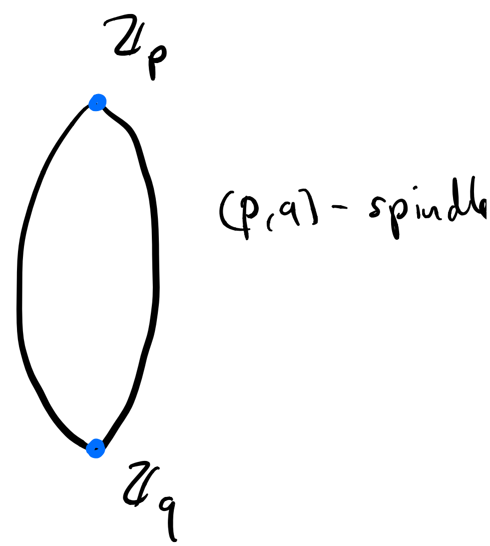
let p, q coprime (e.g. 2 & 3)

Then S^1 acts on S^3 via

$$z(v, w) = (z^p v, z^q w)$$

This action is semi-free:

Pf: $S^1_{(v,w)}$ $\quad z, v \neq 0$



$z \in S^1$ st $(z^p v, z^q w) = (v, w)$

$$z^p = 1 \qquad z^q = 1$$

$$S^1_{(v,0)} = \mathbb{Z}_p$$

$$S^1_{(0,w)} = \mathbb{Z}_q$$

Underlying space is homeomorphic to S^2, but .....

Orbiropes

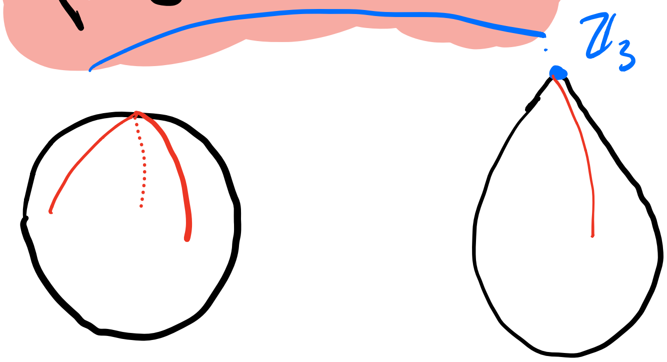
Goal: degree theory for orbifolds

Thm: Classification of $\overbrace{\text{one-dim}}^{\text{connected compact}}$ orbifolds with bdy

$[0,1]$ or S^1 or $[0,1]/\mathbb{Z}_2$ $S^1/\mathbb{Z}_2$.

Orbiropes can have an odd number of ends!

Movie time



$\mathbb{CP}^1 \qquad \mathbb{CP}^1(3,1)$

$$[z_0, z_1] \longmapsto [z_0^3, z_1]$$

Every point on r.h.s. is regular value, but preimage of curve is <u>not</u> an orbifold

not an orbifold

Orbifold degree theory.

Def (Pasquotto - R.) Let $f : P \longrightarrow \mathcal{O}$ be smooth map of orbifolds. Define, for regular y

$$\deg_2(f; y) = \sum_{x \in f^{-1}(y)} \frac{|G_y|}{|G_x|} \mod 2$$

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(d\tilde{f}_{\tilde{x}}) \frac{|G_y|}{|G_x|}$$

under orientation assumptions

- These are not rational numbers.
- Singular points of the orbifold can be regular values
- Without further assumptions degree does depend on y, and is not a homotopy invariant.

Thm Suppose $f : \mathcal{O} \xrightarrow[\text{proper}]{} P$, P connected

and $\mathcal{O}$ does not have codim 1 singularities

* Then $\deg_2(f; y)$ does not depend on y
* Nor on homotopy class of f.
* $\deg_2(f) = 1 \Rightarrow f$ is surjective

+ similar statement for integer degree

$$\boxed{\begin{array}{l}\text{First point is remarkable} \\ \text{orbifold diffeo group is not transitive.}\end{array}}$$

Ex: $z \cdot (z_0, \dots, z_n) = (z^{q_0} z_0, \dots, z^{q_n} z)$

q_i pairwise coprime. $\Rightarrow$ $\mathbb{C}P(q) = S^{2n+1}/_{S^1} \approx \mathbb{C}^{n+1} \setminus \{0\}/_{\mathbb{C}^*}$

$$f_q : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$$

$$f_q(z_0, \dots, z_n) = (z_0^{q_0}, \dots, z_n^{q_n})$$

$\Rightarrow$ $$f_q : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n(q).$$

$$g_q(z_0, \dots, z_n) = \left(z^{\frac{\mathrm{lcm}(q)}{q_0}}, \dots, z^{\frac{\mathrm{lcm}\, q}{q_n}}\right)$$

$\Rightarrow$ $$g_q : \mathbb{C}P^n(q) \longrightarrow \mathbb{C}P^n.$$

$$\deg f_q = q_0 \cdots q_n. \qquad \deg g_q = \frac{(\mathrm{lcm}(q))^n}{q_0 \cdots q_n}.$$

$$h_{rq} : \mathbb{C}P^n(q) \xrightarrow{g_q} \mathbb{C}P^n \xrightarrow{f_r} \mathbb{C}P^n(r)$$

$$\deg h_{rq} = \frac{(\mathrm{lcm}(q))^n}{q_0 \cdots q_n} r_0 \cdots r_n.$$

Some open problems

1) In manifold case degree classifies

$$[M^n, S^n] \xrightarrow{\deg f} \mathbb{Z} \qquad M \text{ oriented}$$

$$[M^n, S^n] \xrightarrow{\deg_2 f} \mathbb{Z}_2 \qquad M \text{ is not oriented}$$

How does this work in orbifold setting?

2) If $m > n$ there is a map

$$[M^m, N^n] \xrightarrow[\text{of regular value}]{\text{preimage}} N_{m-n}(M)$$

cobordism ring

How does this work in orbifold world?

3) The orbifold cobordism ring is trivial. But not with assumptions on singularities.