Formal Stability of Elliptic Equilibria. Applications

Patricia Yanguas
Dpt. de Estadística, Informática y Matemáticas
&
Institute for Advanced Materials and Mathematics
Universidad Pública de Navarra, 31006 Pamplona (Navarra, Spain)

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Vrije Universiteit Amsterdam

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Co-authors

- Jesús F. Palacián, Universidad Pública de Navarra, Spain

- Scott Dumas, University of Cincinnati, USA
- Ken Meyer, University of Cincinnati, USA

- Daniela Cárcamo-Díaz, Universidad del Bío-Bío, Chile
- Claudio Vidal, Universidad del Bío-Bío, Chile

- Manuel Iñarrea, Víctor Lanchares, Ana I. Pascual, J. Pablo Salas, Universidad de La Rioja, Spain

- Fahimeh Mokhtari, Vrije Universiteit Amsterdam, The Netherlands

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Hamiltonian system

A Hamiltonian system is a system of $2n$ ordinary differential equations of the form

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}},$$

where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $H(\mathbf{x}, \mathbf{y})$ is the Hamiltonian function.

We consider $H(-, \mathbf{x}, \mathbf{y})$ to be an autonomous system.

An equilibrium of the Hamiltonian system satisfies the equations

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}} = \mathbf{0}, \qquad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} = \mathbf{0}.$$

The Hamiltonian H can be expanded around the equilibrium as a Taylor series in the form

$$H = H_2 + H_3 + H_4 + \ldots.$$

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Configuration: Relative equilibrium

We consider the linearisation of H around the equilibrium to be in the form:

$$H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \ldots + \frac{\omega_n}{2}(x_n^2 + y_n^2) = \omega_1 I_1 + \ldots + \omega_n I_n,$$

where $\omega_i \in \mathbb{R} \setminus \{0\}$ and $(\mathbf{I}, \phi)$ are the classical action-angle variables:

$$I_i = \frac{\omega_i}{2}(x_i^2 + y_i^2), \qquad \phi_i = \arctan\left(\frac{y_i}{x_i}\right), \qquad i = 1, \ldots, n.$$

The expression in matrix form is: $H_2 = \frac{1}{2}\mathbf{x}^T A\, \mathbf{x}$.

The equilibrium is elliptic when:

- all eigenvalues of A are pure imaginary: $\pm \imath\omega_i$.
- the linearisation matrix A is diagonalisable.

Equilibrium's stability

- An equilibrium is spectrally stable if all eigenvalues of A are pure imaginary.

- An equilibrium is linearly stable when all eigenvalues are pure imaginary and A is diagonalisable.

  All orbits of the tangent flow are bounded for all forward time.

  In particular, an elliptic point is always linearly stable.

- An equilibrium $\mathbf{z}_0$ is Liapunov stable if for every neighbourhood V of $\mathbf{z}_0$, there exists a neighbourhood $U \subseteq V$ such that $\mathbf{z}(0) \in U \Rightarrow \mathbf{z}(t) \in V$ for all forward time.

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Formal integrals and normal forms

A function F is a formal first integral of Hamiltonian H when

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial F}{\partial y_i} \frac{\partial H}{\partial x_i} \right) = 0.$$

Goal

Establish a criterion to obtain formal stability for elliptic equilibria in autonomous Hamiltonian systems with n DOF.

Find positive definite formal integrals of the Hamiltonian.

In case of formal stability, prove that the solutions are bounded near the equilibrium over exponentially long times.

Formal stability versus Lie stability

Definition

We say that the origin of $\mathbb{R}^{2n}$ is formally stable if there exists a formal series which is positive definite near the origin and a formal integral for the full Hamiltonian.

Definition

We say that the origin of $\mathbb{R}^{2n}$ is Lie stable if there exists $p \geq 2$ such that the truncated Hamiltonian function in normal form associated to $\mathcal{H}^m$ is stable (in the sense of Liapunov) for any (arbitrary) $m \geq p$.

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Lie stability features

- Lie stability for elliptic equilibria $\Rightarrow$ formal stability.

- Formal stability theory was initiated by Siegel (1954), Moser (1955), Glimm (1964), Bryuno (1967), etc.

- Khazin (1971) introduced the concept of Lie stability, although he named it as “Birkhoff stability” in the case of elliptic equilibria.

- dos Santos and Vidal developed the concept of Lie stability for resonant systems.

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Key elements for Lie stability

Exploit the algebraic structure of the linear part
of the equation as much as we can.

$$H_2(\mathbf{I}) = \sum_{k=1}^{d} \sigma_k F_k(\mathbf{I}),$$ where the $\sigma_k \neq 0$ are linear combinations of the ω_j,

$\mathbf{I} = (I_1, \ldots, I_n)$ and F_i are first integrals of H_2.

$$S = \{\mathbf{I} \,|\, I_j \geq 0,\ F_1(\mathbf{I}) = \ldots = F_d(\mathbf{I}) = 0\}$$

- n is the number of DOF of the system.
- d is the number of independent first integrals of H_2.
- $s = n - d$.
- $0 \leq \dim S \leq s$.

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Lie stability Theorem

Theorem

D. Cárcamo-Díaz, J.F. Palacián, C. Vidal, P.Y.

Consider $\mathbf{I} \in S$*:*

(A) Suppose there is an even integer j *(with* $4 \leq j \leq p$*) such that* $\mathcal{H}^j(\mathbf{I}, \phi) \neq 0$ *for* $|\mathbf{I}|$ *small enough and all* ϕ*. Then the origin of* $\mathbb{R}^{2n}$ *is Lie stable for the Hamiltonian function* H*.*

(B) Suppose there is an integer $i \geq 3$ *such that* $\mathcal{H}^i(\mathbf{I}, \phi)$ *changes sign for some* $\mathbf{I}$ *and* ϕ*. Then there is no index* j *(with* $i < j \leq p$*) such that* $\mathcal{H}^j(\mathbf{I}, \phi) \neq 0$ *for* $|\mathbf{I}|$ *sufficiently small.*

- The case $n = 2$ corresponds to Cabral & Meyer stability result.
- The lower $\dim S$ is, the more cases of Lie stable systems we get.
- When $S = \{\mathbf{0}\}$ there is always Lie stability.

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Estimates

- It is said that vector $\sigma = (\sigma_1, \ldots, \sigma_d)$ satisfies a *Diophantine condition* when there are fixed constants $c > 0$ and $\nu > d - 1$ such that

$$\forall\, \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\,, \quad |\mathbf{k} \cdot \sigma| \geq c|\mathbf{k}|^{-\nu}\,. \tag{1}$$

Theorem

D.Cárcamo-Díaz, J.F. Palacián, C. Vidal, P.Y.

If the real analytic Hamiltonian H has the origin of $\mathbb{R}^{2n}$ as a formally stable equilibrium according to hypotheses (A) of the previous theorem, while the frequency vector σ satisfies the Diophantine condition (1), then there exist $C > 0$, $K > 0$, $a > 1$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, and for all $\mathbf{x}_0$ with $|\mathbf{x}_0| < \varepsilon$ we have

$$|\mathbf{x}(t, \mathbf{x}_0)| < a\, \varepsilon^{2/j} \quad \textit{for all} \;\; t \;\; \textit{with} \quad 0 \leq t \leq T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right)\,.$$

Estimates behaviour

- The fewer terms are needed to conclude Lie stability (j), the better the bounds on the solutions are.

- The fewer independent formal integrals (d) there are, the larger time T can be.

- When $d = 1$ the Diophantine condition is not required.

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Formal stability versus Nekhoroshev estimates

Niederman, Bennetin, Fassò, Guzzo, Pöschel,... ($\sim$ 1998).

- $\mathcal{H}^j(\mathbf{I})$ is convex at $\mathbf{I} = \mathbf{0}$ if the quadratic form $\mathcal{H}_4(\mathbf{I})$ is definite.
- It is quasi-convex at $\mathbf{I} = \mathbf{0}$ if $H_2(\mathbf{I}) = \mathcal{H}_4(\mathbf{I}) = 0 \Rightarrow \mathbf{I} = \mathbf{0}$;
  It corresponds to Glimm's criterion on formal stability.
- It is directionally quasi-convex at $\mathbf{I} = \mathbf{0}$ if H_2 and $\mathcal{H}_4$ vanish simultaneously for $I_i \geq 0$ only at $\mathbf{I} = \mathbf{0}$;
  It is equivalent to Bryuno's hypotheses for formal stability (1967);
  It is a particular case of Lie stability.
- Glimm's ideas inspired Nekhoroshev to establish the concept of steepness of a function.
- Convexity on S.

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Pros and cons

We can get Lie stability for Hamiltonians that do not satisfy the conditions needed in Nekhoroshev theory:

- When the formal stability is deduced from Hamiltonians depending on resonant angles.
- When the normal form is too degenerate to establish a certain convexity condition.

There are steep systems for which we cannot conclude Lie stability.

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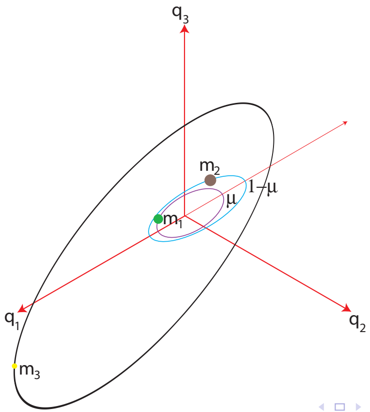
   - Nonlinear stability of the attitude of a satellite describing a circular orbit in space
   - Nonlinear stability of the Levitron

The system

Motion in the $3D$ space of an infinitesimal particle under the gravitational attraction of m_1 and m_2 that describe circular orbits around their common centre of mass.



The Hamiltonian

- Hamiltonian in rectangular coordinates (x, y, z, X, Y, Z) in a rotating reference frame:

$$\mathcal{H} = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) + Xy - xY - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} - \frac{1-\mu}{\sqrt{(\mu+x)^2 + y^2 + z^2}}.$$

- $\mu = \frac{m_2}{m_1+m_2}$.
- Assuming $m_1 \geq m_2$, then $\mu \in (0, 1/2]$.
- m_1 is located at $(-\mu, 0, 0)$ and m_2 is at $(1-\mu, 0, 0)$.

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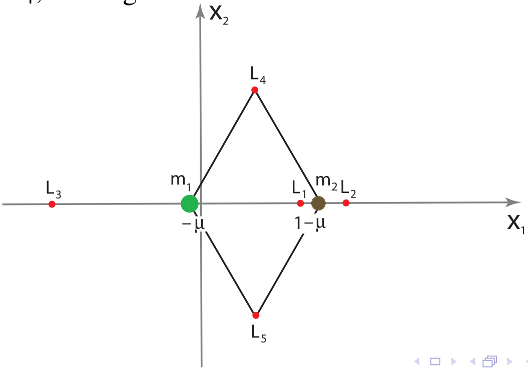
The Lagrange equilibria

- The stability of L_4 and L_5 depends on μ.
- Coordinates of L_4 and L_5: $\left(\frac{1}{2}-\mu,\ \pm\frac{\sqrt{3}}{2},\ 0,\ \mp\frac{\sqrt{3}}{2},\ \frac{1}{2}-\mu,\ 0\right)$, where the upper sign applies for L_4 and the lower sign does for L_5.
- The stability of both equilibria is the same, so from now on we only refer to the point L_4, although the same conclusions are valid for L_5.



Taylor expansion

We translate the equilibrium solution L_4 to the origin by means of:

$$x = x_1 + \frac{1}{2} - \mu, \quad y = y_1 + \frac{\sqrt{3}}{2}, \qquad z = z_1,$$

$$X = X_1 - \frac{\sqrt{3}}{2}, \qquad Y = Y_1 + \frac{1}{2} - \mu, \quad Z = Z_1.$$

Then, the Hamiltonian function is expanded in Taylor series around the origin, constant terms are eliminated and we get a Hamiltonian of the form

$$H = H_2 + H_3 + \cdots + H_j + \cdots .$$

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The Routh critical value

Eigenvalues of the linearization:

$$\pm\lambda_1, \pm\lambda_2, \pm\lambda_3.$$

$$\lambda_1 = i\omega_1,\ \lambda_2 = i\omega_2,\ \lambda_3 = i\omega_3 \quad \Leftrightarrow \quad 0 < \mu < \mu_R = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right)$$

$$\omega_1 = \frac{\sqrt{1 + \sqrt{1 - 27\mu + 27\mu^2}}}{\sqrt{2}},\ \omega_2 = \frac{\sqrt{1 - \sqrt{1 - 27\mu + 27\mu^2}}}{\sqrt{2}},\ \omega_3 = 1.$$

$$0 < \omega_2 < \frac{\sqrt{2}}{2} < \omega_1 < 1 \quad \text{and} \quad \omega_1^2 + \omega_2^2 = 1.$$

$$\mu \in (0, \mu_R) \Leftrightarrow \omega_1 \in \left(\frac{\sqrt{2}}{2}, 1\right)$$

The normal form in action-angle variables

$$H = H_2 + \mathcal{H}_4 + \cdots ,$$

where

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3,$$

$$\mathcal{H}_4 = c_{200} I_1^2 + c_{110} I_1 I_2 + c_{101} I_1 I_3 + c_{020} I_2^2 + c_{011} I_2 I_3 + c_{002} I_3^2$$

$$c_{200} = \frac{\omega_2^2(124\omega_1^4 - 696\omega_1^2 + 81)}{144(1-2\omega_1^2)^2(1-5\omega_1^2)}, \quad c_{110} = -\frac{\omega_1\omega_2(64\omega_1^2\omega_2^2+43)}{6(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)},$$

$$c_{101} = \frac{-8\omega_1\omega_2^2}{3(1-2\omega_1^2)(4-\omega_1^2)}, \quad c_{020} = \frac{\omega_1^2(124\omega_1^4 + 448\omega_1^2 - 491)}{144(1-2\omega_1^2)^2(1-5\omega_2^2)},$$

$$c_{011} = \frac{8\omega_2\omega_1^2}{3(1-2\omega_2^2)(4-\omega_2^2)}, \quad c_{002} = \frac{-\omega_1^2\omega_2^2}{3(4-\omega_1^2)(4-\omega_2^2)}.$$

Resonances

- The system presents a resonance relation if there exists an integer vector $\mathbf{k} = (k_1, k_2, k_3) \neq \mathbf{0}$ such that

$$k_1\omega_1 - k_2\omega_2 + k_3\omega_3 = 0.$$

  Vector $\mathbf{k}$ is known as the resonance vector and vector $\omega = (\omega_1, \omega_2, \omega_3)$ is the frequency vector.

- Consider the frequency vector $\omega = (m/n, \sqrt{n^2 - m^2}/n, 1)$, with $m, n \in \mathbb{Z}^+$ and $0 < m < n$. Vector $(m, \sqrt{n^2 - m^2}, n)$ is a Pythagorean triple if and only if $n^2 - m^2$ is a perfect square or, equivalently, $\omega_2 \in \mathbb{Q}$. In this case, we will say that vector ω is associated with a Pythagorean triple.

General case

$$\omega_1 I_1 - \omega_2 I_2 + I_3 = 0;\ k_1\omega_1 - k_2\omega_2 + k_3 = 0;\ k_1, k_2, k_3 \in \mathbb{Z};\ I_1, I_2, I_3 \geq 0.$$

Case	ω_1	ω_2	F_i	d	s	S	dim S
(a_1)	$\mathbb{Q}$	$\mathbb{Q}$	$F = \omega_1 I_1 - \omega_2 I_2 + I_3$	1	2	$\left\{\left(I_1, \frac{I_3+\omega_1 I_1}{\omega_2}, I_3\right) \mid I_1, I_3 \geq 0\right\}$	2
(a_2)	$\mathbb{Q}$	$\mathbb{R}\setminus\mathbb{Q}$	$F_1 = \omega_1 I_1 + I_3$ $F_2 = I_2$	2	1	$\{\mathbf{0}\}$	0
(b_1)	$\mathbb{R}\setminus\mathbb{Q}$	$\mathbb{R}\setminus\mathbb{Q}$	$F_1 = I_1$ $F_2 = I_2$ $F_3 = I_3$	3	0	$\{\mathbf{0}\}$	0
(b_2)	$\mathbb{R}\setminus\mathbb{Q}$	$\mathbb{R}\setminus\mathbb{Q}$	$F_1 = \frac{k_2}{k_1} I_1 - I_2$ $F_2 = -\frac{k_3}{k_1} I_1 + I_3$	2	1	$\left\{\left(I_1, \frac{k_2}{k_1} I_1, \frac{k_3}{k_1} I_1\right) \mid I_1 \geq 0\right\}$ $\{\mathbf{0}\}$	1 0
(b_3)	$\mathbb{R}\setminus\mathbb{Q}$	$\mathbb{Q}$	$F_1 = I_1$ $F_2 = -\omega_2 I_2 + I_3$	2	1	$\{(0, I_3/\omega_2, I_3) \mid I_3 \geq 0\}$	1

Case (a_1)

ω_1	ω_2	F_i	d	s	S	dim S
$\mathbb{Q}$	$\mathbb{Q}$	$F = \omega_1 I_1 - \omega_2 I_2 + I_3$	1	2	$\left\{\left(I_1, \frac{I_3+\omega_1 I_1}{\omega_2}, I_3\right) \mid I_1, I_3 \geq 0\right\}$	2

$$\mathcal{H}^4(\mathbf{I}) = \beta_1 I_1^2 + \beta_2 I_1 I_3 + \beta_3 I_3^2$$

with

$$\beta_1 = \frac{644\omega_1^8 - 1288\omega_1^6 + 1185\omega_1^4 - 541\omega_1^2 + 36}{16(1-\omega_1^2)(1-2\omega_1^2)^2(1-5\omega_1^2)(4-5\omega_1^2)},$$

$$\beta_2 = \frac{\omega_1\left(18580\omega_1^{12} - 67928\omega_1^{10} + 70827\omega_1^8 + 30890\omega_1^6 - 62113\omega_1^4 + 22128\omega_1^2 - 8496\right)}{72(1-\omega_1^2)\left(1-2\omega_1^2\right)^2\left(1-5\omega_1^2\right)\left(3+\omega_1^2\right)\left(4-\omega_1^2\right)\left(4-5\omega_1^2\right)},$$

$$\beta_3 = \frac{\omega_1^2\left(960\omega_1^{10} - 7364\omega_1^8 + 29940\omega_1^6 - 48219\omega_1^4 + 24155\omega_1^2 - 444\right)}{144(1-\omega_1^2)\left(1-2\omega_1^2\right)^2(3+\omega_1^2)(4-\omega_1^2)(4-5\omega_1^2)}.$$

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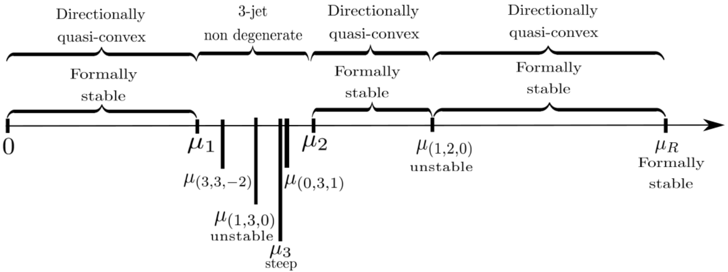
Summary

Theorem

For $0 < \mu < \mu_R$ *the equilibrium point* L_4 *is Lie stable, excepting the unstable cases* $\mu_{(1,2,0)}$, $\mu_{(1,3,0)}$ *and the values* $\mu \in (\mu_1, \mu_2)$ *leading to a Pythagorean triple.*



Estimates

$$|\mathbf{I}(t)| < \alpha\, \epsilon^{2/j} \quad \text{for all} \quad t \quad \text{with} \quad 0 \leq t \leq T = \mathcal{C} \exp\left(\frac{\mathcal{E}}{\epsilon^{1/(2(\nu+1))}}\right)$$

1. $S = \{\mathbf{0}\} \Rightarrow j = 2, d = 2$ or $d = 3 \Rightarrow \nu \geq 1$ or $\nu \geq 2$.
2. $S \neq \{\mathbf{0}\} \Rightarrow j = 4, d = 2$ or $d = 1 \Rightarrow \nu \geq 1$ or $\nu \geq 0$.

Bounds are sharper when Lie stability is obtained through a low-order normal form and low number of integrals.

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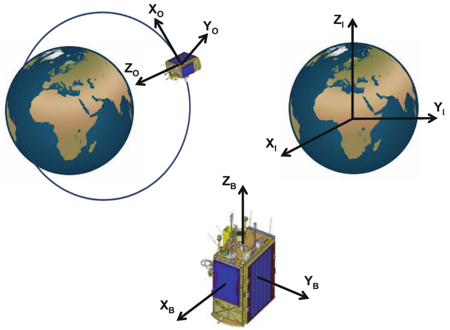
NONLINEAR STABILITY OF THE ATTITUDE OF A SATELLITE DESCRIBING A CIRCULAR ORBIT IN SPACE

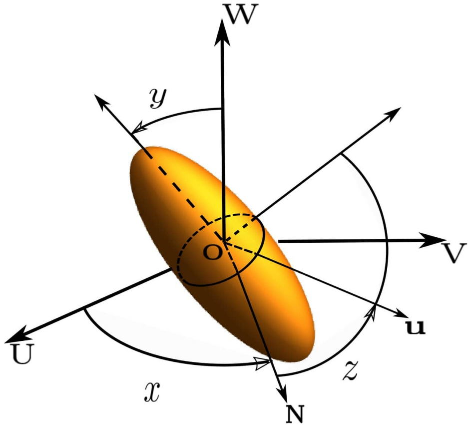


Taken from: Stability, Pointing, and Orientation, Willem Herman Steyn, in J. N. Pelton (ed.), Handbook of Small Satellites, Springer Nature Switzerland AG 2020.

A.P. Markeev, A.G. Sokol'skii, On the stability of relative equilibrium of a satellite in a circular orbit, Kosmicheskie Issledovaniya, 13(2), 139–146 (1975); Cosm. Res., 13(2), 119–125 (1975).

The system

$H(x, y, z, X, Y, Z; A, C)$ with $A = \frac{a}{b}$, $C = \frac{c}{b}$ and a, b, c being the principal central moments of inertia.



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The equilibria

- There are 24 equilibria: $P_1, \dots, P_{24}$.
- The system enjoys 4 independent discrete symmetries: $S_1, \dots, S_4$.

$$\begin{cases} P_1 \\ P_5 \\ \vdots \\ P_{11} \end{cases} \xrightarrow{S_4} \begin{cases} P_1 \\ P_6 \\ P_8 \\ P_{10} \end{cases} \xrightarrow{S_1} \begin{cases} P_1 \\ P_8 \end{cases} \xrightarrow{S_2} P_1$$

$$\begin{cases} P_2 \\ P_{12} \\ \vdots \\ P_{18} \end{cases} \xrightarrow{S_4} \begin{cases} P_2 \\ P_{13} \\ P_{15} \\ P_{17} \end{cases} \xrightarrow{S_1} \begin{cases} P_2 \\ P_{13} \end{cases} \xrightarrow{S_3} P_2$$

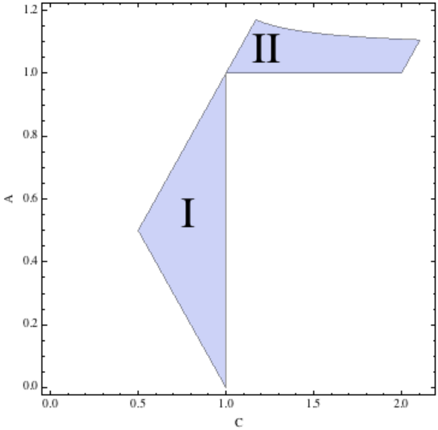
$$\begin{cases} P_3 \\ P_{19} \\ P_{20} \\ P_{21} \end{cases} \xrightarrow{S_4} \begin{cases} P_3 \\ P_{20} \end{cases} \xrightarrow{S_1} P_3$$

$$\begin{cases} P_4 \\ P_{22} \\ P_{23} \\ P_{24} \end{cases} \xrightarrow{S_4} \begin{cases} P_4 \\ P_{23} \end{cases} \xrightarrow{S_1} P_4$$

Necessary conditions for linear stability of the equilibria

I: Lagrange region:
Liapunov stability

II: Beletskii-DeBra-Delp region



(Beletskii, 1960, 1966)
(Delp, 1958) and (DeBra, Delp, 1961).

The normal form in action-angle variables

$$H(\mathbf{I},\theta) = H_2(\mathbf{I}) + \mathcal{H}_4(\mathbf{I}) + \cdots ,$$

where

$$H_2(\mathbf{I}) = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3,$$

in region I,

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3,$$

in region II and

$$\mathcal{H}_4(\mathbf{I}) = c_{200} I_1^2 + c_{020} I_2^2 + c_{002} I_3^2 + c_{110} I_1 I_2 + c_{011} I_2 I_3 + c_{101} I_1 I_3.$$

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Formal integrals and set S

- $H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3$
- $\overline{\omega}_1 = \frac{\omega_1}{\omega_3}$, $\overline{\omega}_2 = \frac{\omega_2}{\omega_3}$, $\overline{\omega}_3 = 1$.
- The number of linearly independent integrals is $1 \leq d(= 3 - s) \leq 3$.
- When $\mathbf{I} \in S$ then $H_2(\mathbf{I}) = 0$ and

$$I_2 = \frac{1}{\overline{\omega}_2} \left(\overline{\omega}_1 I_1 + I_3 \right),$$

  with $I_1, I_3 \geq 0$.
- $k_1 \overline{\omega}_1 - k_2 \overline{\omega}_2 + k_3 = 0$, with $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$.
- $S = \{\mathbf{0}\} \implies$ Lie stability holds from H_2.

Cases

(a_1) $\overline{\omega}_1, \overline{\omega}_2 \in \mathbb{Q}$.

(a_2) $\overline{\omega}_1 \in \mathbb{Q}$ and $\overline{\omega}_2 \in \mathbb{R} \setminus \mathbb{Q}$.

(b_1) $\overline{\omega}_1, \overline{\omega}_2 \in \mathbb{R} \setminus \mathbb{Q}$ and there are no resonances among the I_j.

(b_2) $\overline{\omega}_1, \overline{\omega}_2 \in \mathbb{R} \setminus \mathbb{Q}$ and there is an integer vector $\mathbf{k} \neq \mathbf{0}$ such that

$$\overline{\omega}_1 = \frac{k_2}{k_1}\overline{\omega}_2 - \frac{k_3}{k_1}.$$

(b_3) $\overline{\omega}_1 \in \mathbb{R} \setminus \mathbb{Q}$ and $\overline{\omega}_2 \in \mathbb{Q}$.

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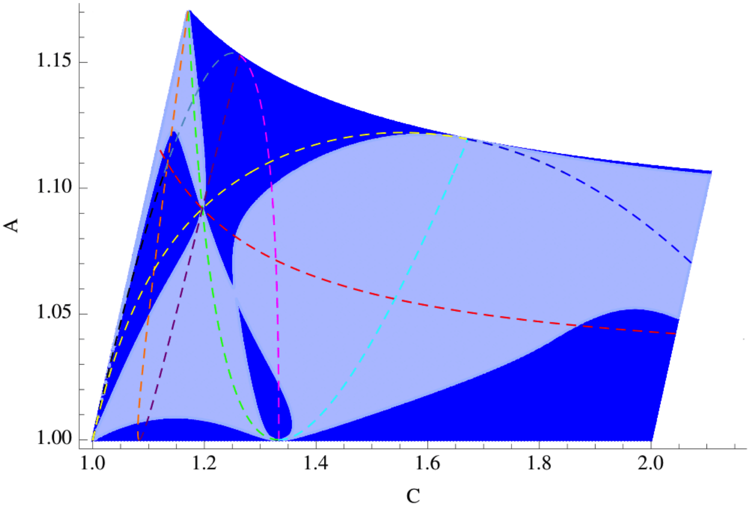
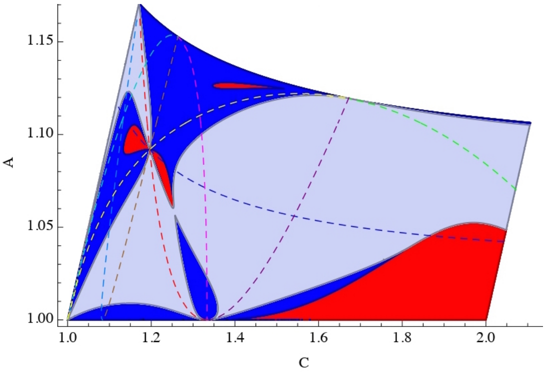
Lie stability regions



Lie versus Nekhoroshev

6. The union of the regions of quasi-convexity and directional quasi-convexity corresponds to the region where there is Lie stability for any value of the frequencies (excluding the resonance lines).



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What is a Levitron?

The Levitron top is a device commercialised as a toy that displays the phenomenon known as “spin-stabilized magnetic levitation”.

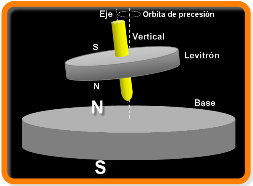
The model

Two reference frames:

(i) An inertial frame attached to the base, centered at its centre: coordinates of the c.m. of the top w.r.t. this frame are (x, y, z).

(ii) A non-inertial frame attached to the spinning top: Euler's angles ϑ, φ, ψ give the top's orientation w.r.t. the inertial frame.

- Two forces: magnetic field created by the magnetic spinning top and the repelling base magnet and gravity of the top.
- The top is axi-symmetric and has inertia tensor $\operatorname{diag}\{\Theta_1, \Theta_1, \Theta_3\}$.



Equations of motion

The total energy is conserved:

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2\Theta_1}\left(p_\vartheta^2 + \frac{(p_\varphi - p_\psi \sin\vartheta)^2}{\cos^2\vartheta}\right) + \frac{p_\vartheta^2}{2\Theta_3} + U(r, \vartheta, \varphi)$$

with potential

$$\begin{aligned} U(r, \vartheta, \varphi) &= mgz - \mu\left(\tfrac{1}{2}\Phi_2(z)(xR_{13} + yR_{23})\right. \\ &\qquad \left. + (-\Phi_1(z) + \tfrac{1}{4}(x^2 + y^2)\Phi_3(z))R_{33} + \dots\right), \end{aligned}$$

where $R = (R_{ij})$ is an orthogonal 3×3-matrix and

$$\Phi_k(z) = \frac{d^k}{dz^k}V_0(z), \qquad V_0(z) = 2\pi z\left(\frac{1}{\sqrt{W^2 + z^2}} - \frac{1}{z}\right).$$

p_ψ is an integral of the Hamiltonian system related to $\mathcal{H}$.

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Equilibrium

We require that the gravity and the magnetic forces compensate each other, preventing the device to be pushed downwards (gravity dominates) or upwards (magnetic force dominates).

In 6 DOF we get a periodic solution with coordinates

$$(0\,,\,0\,,\,z_s\,,\,\sigma t\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,\sigma\Theta_3\,,\,0\,,\,0) \quad \text{with } \sigma = (\mu\Phi_3(z_s)/m)^{1/2},$$

and z_s is the real solution of

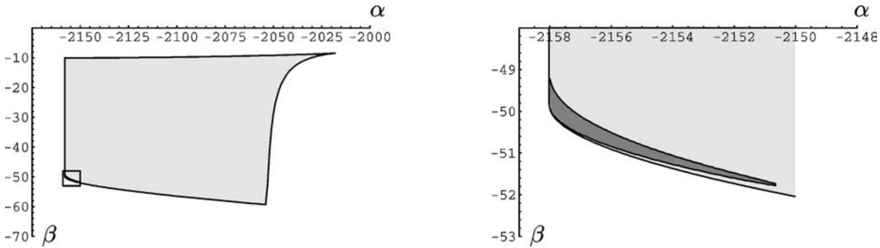
$$-C_1 - \frac{6\pi C_2 W^2 z}{(W^2+z^2)^{5/2}} = 0,$$

with

$$C_1 = \alpha\beta\gamma, \quad C_2 = \frac{\beta}{\gamma}, \quad \alpha = m\frac{g}{\mu}, \quad \beta = m\mu, \quad \gamma = \sqrt{\frac{\Theta_3}{m}}.$$

Non-linear stability

Dullin and Fassò apply Nekhoroshev theory:



The region of non-linear stability (dark grey) is very small compared to the region of linear stability.

Experiments show stability in a much wider region.

The normal form in action-angle variables #1

Define $I_i = (x_i^2 + y_i^2)/2, \quad \phi_i = \arctan(y_i/x_i)$.

$$H(\mathbf{I}, \phi) = H_2(\mathbf{I}) + \mathcal{H}_4(\mathbf{I}) + \cdots ,$$

where

$$H_2 = -\omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 + \omega_4 I_4 + \omega_5 I_5,$$

or

$$H_2 = -\omega_1 I_1 + \omega_3 I_3 + \omega_4 I_4 + \omega_5 I_5,$$

or

$$H_2 = -\omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3 + \omega_4 I_4 + \omega_5 I_5.$$

$\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 > 0$

General case

- We have obtained all possibilities for S between $\{0\}$ and $\dim S = 4$.
- There can be resonant cases with $\dim S = 0$.
- The most generic situation is $S = \{0\}$. Thus, Lie stability extends with positive measure to the whole region of linear stability.

Final

THANK YOU FOR YOUR ATTENTION!