A brief history of reduction types of algebraic curves

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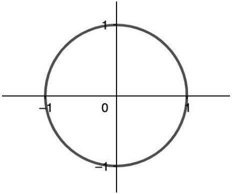
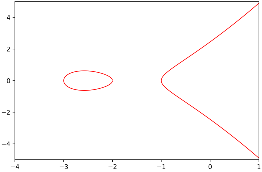
For example, the following are algebraic curves over $\mathbb{R}$:



$x^2 + y^2 = 1$



$y^2 = (x+1)(x+2)(x+3)$

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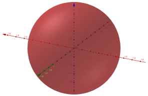
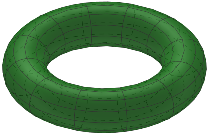
...while these are the "same" curves, over $\mathbb{C}$[1]:





[1]Complex curves are Riemann surfaces.

A first numerical invariant: the genus

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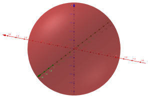
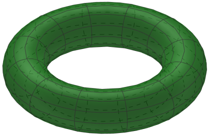


Genus 0



Genus 1

A first numerical invariant: the genus

Definition

~~The genus of a curve is the dimension of the space of regular differential 1-forms on it.~~(For smooth curves), the genus is the number of "holes" that the curve has when seen over $\mathbb{C}$.

Example

The genus-degree formula tells us that if our curve is an irreducible plane curve given by a homogeneous polynomial of degree d, then the genus is:

$$g = \frac{(d-1)(d-2)}{2}.$$

Genus 0 curves

Curves of genus 0 are "easy" to understand. Let X be such a curve, then:

- X is given by a polynomial of degree 1 or 2 (so X is either a line or a conic).
- Over $\mathbb{C}$, X is a sphere (equivalently, a projective line).
- Over a number field K, if X has a K-rational point, it is the projective line.

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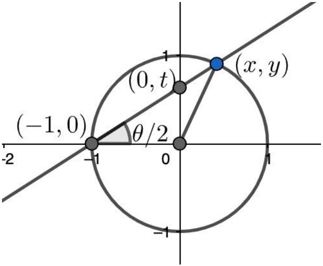
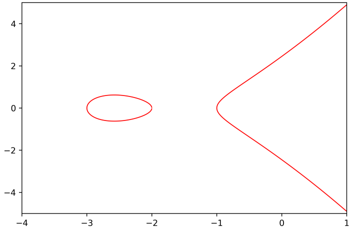


A rational parametrization of the unit circle: $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$.

Genus 1 curves

Genus 1 curves (which are smooth, projective and have at least one rational point) are **elliptic curves**. If K is a number field, an elliptic curve E over K can be expressed via a Weierstrass equation, i.e. an equation of the form $y^2 = f(x)$, where $f(x) \in K[x]$ has degree 3. The point at infinity is always a rational point.



$E : y^2 = (x+1)(x+2)(x+3)$ is an elliptic curve.

Reduction types of elliptic curves – I

Let E be a curve given by a Weierstrass equation over a number field K. Write:

$$E : y^2 = x^3 + ax + b \qquad a, b \in K.$$

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Let $\Delta = -16(4a^3 + 27b^2)$ be the **discriminant** of the curve, then $\Delta \neq 0$ if and only if $x^3 + ax + b$ has three different roots over an algebraic closure of K, if and only if E is smooth.

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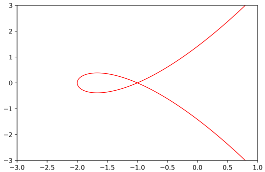
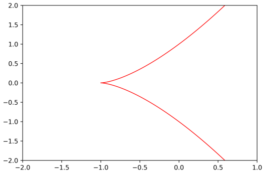


Nodal curve: two roots coincide ($a \neq 0$).



Cuspidal curve: the three roots coincide ($a = 0$).

Reduction types of elliptic curves – II

Let $\mathfrak{p}$ be a prime of K (not dividing $2, 3$), and let $v_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic valuation. Fix a Weierstrass equation for E where $v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b) \geq 0$ and $v_{\mathfrak{p}}(\Delta)$ is minimal.

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over the residue field of K at $\mathfrak{p}$, where $\bar{\cdot}$ is the reduction modulo $\mathfrak{p}$ map.

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Definition

We say E has good reduction if the reduced curve is smooth, multiplicative reduction if it has a node, additive reduction if it has a cusp.

Reduction types of elliptic curves – III

Theorem (Tate '75)

- *E has good reduction at* $\mathfrak{p}$ *iff* $\mathfrak{p} \nmid \Delta$.
- *E has bad multiplicative reduction iff* $\mathfrak{p} \mid \Delta$ *and* $\mathfrak{p} \nmid a$.
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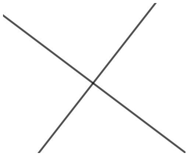
The j-**invariant** of E is $j = 1728 \dfrac{4a^3}{4a^3 + 27b^2}$.

Fact

E has potentially good reduction if and only if $v_{\mathfrak{p}}(j) \geq 0$.

Stable reduction: a definition

We say that an algebraic curve has **semistable** reduction if its only singular points are ordinary double, i.e.



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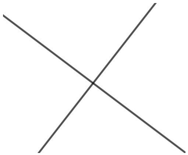


Among the semistable reduction models of a curve, we call stable model one such that each irreducible component of genus 0 intersects the rest of the reduction in at least three points.

Curves of genus > 1

Algebraic curves of genus greater than 1 are either:

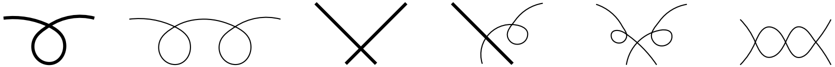
- **hyperelliptic**, i.e. 2-sheet covers of the projective line with $2g + 2$ ramified points (possibly including the point at infinity): these are given by $y^2 = f(x)$ with $\deg(f) \in \{2g + 1, 2g + 2\}$. Or:
- **non-hyperelliptic**, in which case they are embedded into the $(g - 1)$-dimensional projective space.

Theorem (Stable Reduction Theorem, Deligne and Mumford '69)

Every curve of genus $g > 1$ admits a unique stable model over a finite extension of the field of definition.

Genus 2 curves

Genus 2 curves are always hyperelliptic, thus given by $C: y^2 = f(x)$, $\deg(f) \in \{5, 6\}$. The stable reduction types of such curves are 7, namely good reduction and the following bad types:



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The role of the j-invariant is played by the four **Igusa** invariants $I_2,\ I_4,\ I_6, I_{10} = \Delta$.

Theorem (Liu '93)

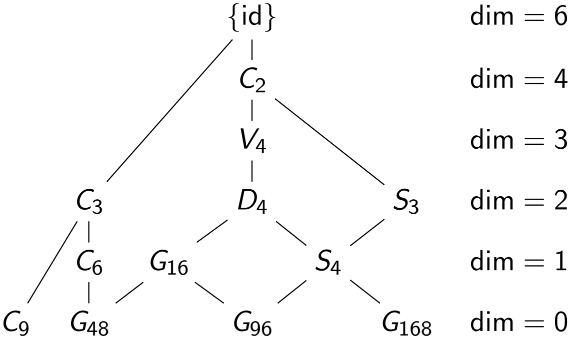
- *C has good reduction iff $\mathfrak{p} \nmid \Delta$.*
- *C has potentially good reduction iff $i\, v_{\mathfrak{p}}(I_{10}) \leq 5\, v_{\mathfrak{p}}(I_{2i})$.*
- *The valuations of $I_2, \ldots, I_{10}$ determine the type of the reduction.*

Genus 3 curves

Genus 3 curves are either hyperelliptic or plane quartics.

- The 9 invariants associated to hyperelliptic curves are called Shioda invariants. Reduction types can be determined using the ramification points, in terms of these invariants (by work of Favereau, based on Dokchitser-Dokchitser-Maistret-Morgan).
- The 13 invariants associated to plane quartics are the Dixmier-Ohno invariants. One way to classify these curves is in terms of their automorphism group.

Automorphisms of a plane quartic



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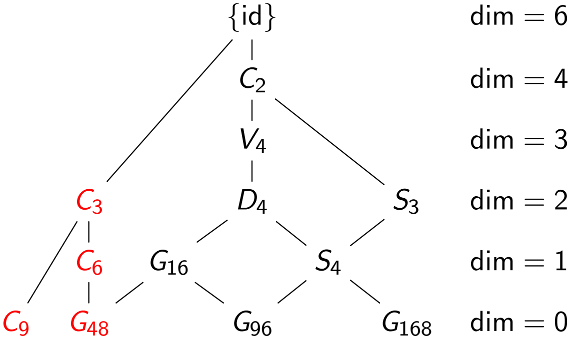

Picard curves

These are Picard curves: $y^3 = f(x)$, where $\deg(f) = 4$.

This is solved by the work about the reduction of superelliptic curves by Bouw and Wewers, that is, for curves of the shape

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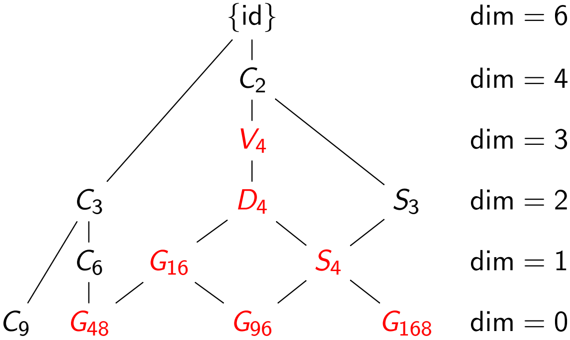
The main idea for studying hyperelliptic and superelliptic curves is to use a Galois cover $C \to \mathbb{P}^1$ and study the reduction of the ramification points.

Automorphisms of a plane quartic



Ciani quartics

A plane quartic Y with $\mathrm{Aut}(Y) \supseteq V_4$ admits a model of the form:

$$Y: \quad Ax^4 + By^4 + Cz^4 + ay^2z^2 + bx^2z^2 + cx^2y^2 = 0.$$

Here, the elements of V_4 act on Y as

$$(x : y : z) \mapsto (\pm x : \pm y : z),$$

and the cover $f : Y \to X = Y/V_4$ is Galois, with 6 ramification points.

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The ring of invariants of such curves Y is generated by

$$\begin{aligned} I_3 &= ABC, & I_3'' &= \Delta(X), \\ I_3' &= A\Delta_a + B\Delta_b + C\Delta_c, & I_6 &= \Delta_a\Delta_b\Delta_c, \end{aligned}$$

with $\Delta_a = a^2 - 4BC$, $\Delta_b = b^2 - 4AC, \Delta_c = c^2 - 4AB$.

Stable reduction for covers

The Stable Reduction Theorem has a Galois covers analogue. It states that there exists a unique minimal semistable model of the marked curve X (markings are ramification points), and its special fiber $\overline{X}$ is a tree of projective lines. Every irreducible component of $\overline{X}$ contains at least 3 points which are either marked or singular points of $\overline{X}$.

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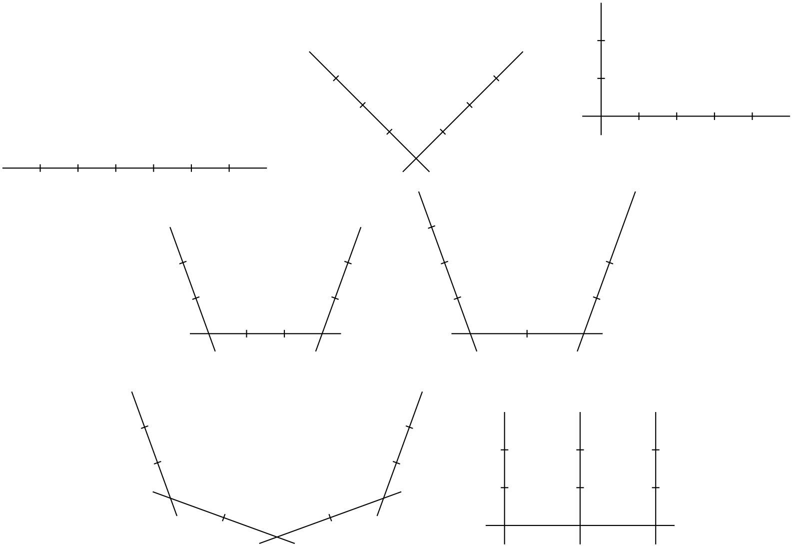
Strategy

- Determine all the possibilities for $\overline{X}$.
- Use "reverse-engineering" to determine the corresponding stable reductions.
- Make this explicit to classify stable reduction types in terms of I_3, I_3', I_3'', I_6.

Possible graphs of $\overline{X}$



Step 1: combinatorial conditions

The action of V_4 on the ramification points is represented by an "acceptable labeling" on the marked curve $\overline{X}$. For every such labeling, there exists a unique cover $\overline{f} : \overline{Y} \to \overline{X}$ and it determines the stable reduction of Y.

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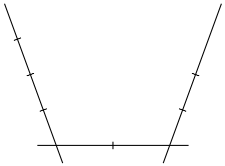
Let D be the set of marked points and S be the set of singular points on $\overline{X}$. Let $\sigma_1, \sigma_2, \sigma_3$ denote the non-trivial elements of V_4. Then, a labeling $I : \overline{D} \cup S \to \{\mathrm{id}, 1, 2, 3\}$ satisfies:

- $\# I^{-1}(i) \cap \overline{D} = 2$ for each i.
- On every component X_i : $\prod_{x \in X_i} \sigma_{I(x)} = \mathrm{id}$.

An example



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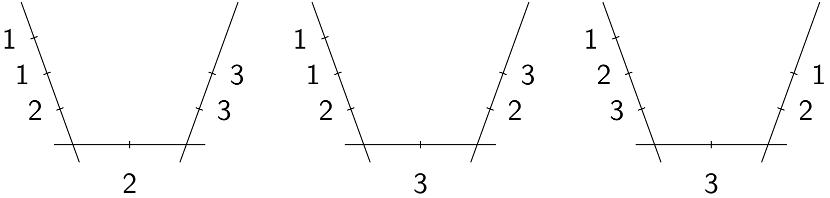

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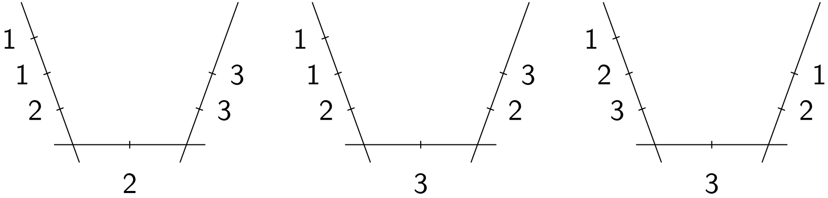
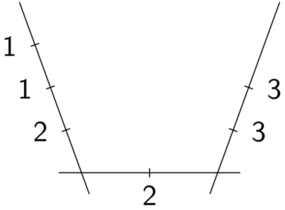


Fact

There are 20 such decorated graphs.

Step 2: reverse engineering (and combinatorics)

We compute the stable reduction of Y from the labeling of $\overline{X}$ as follows.



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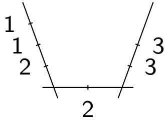

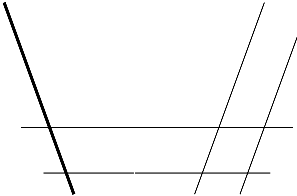

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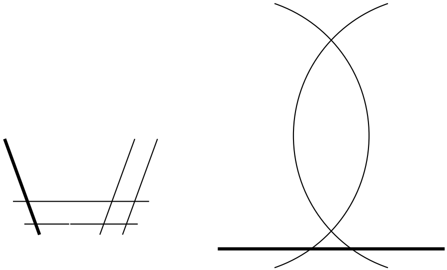

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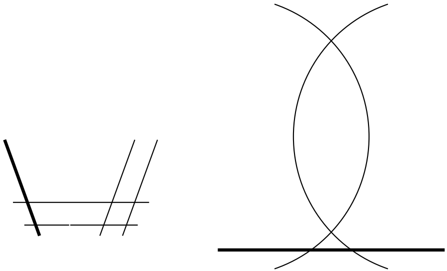
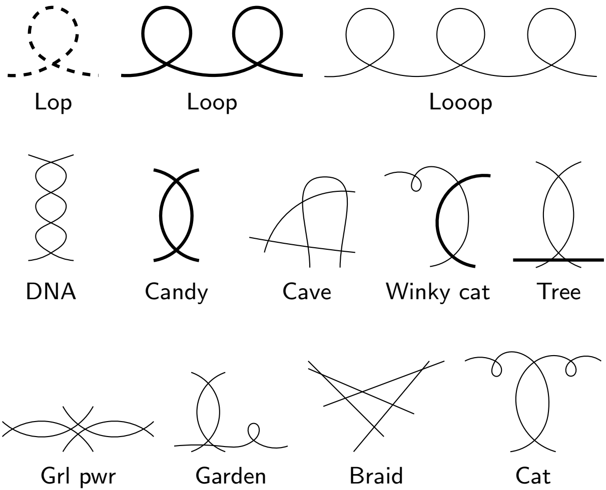
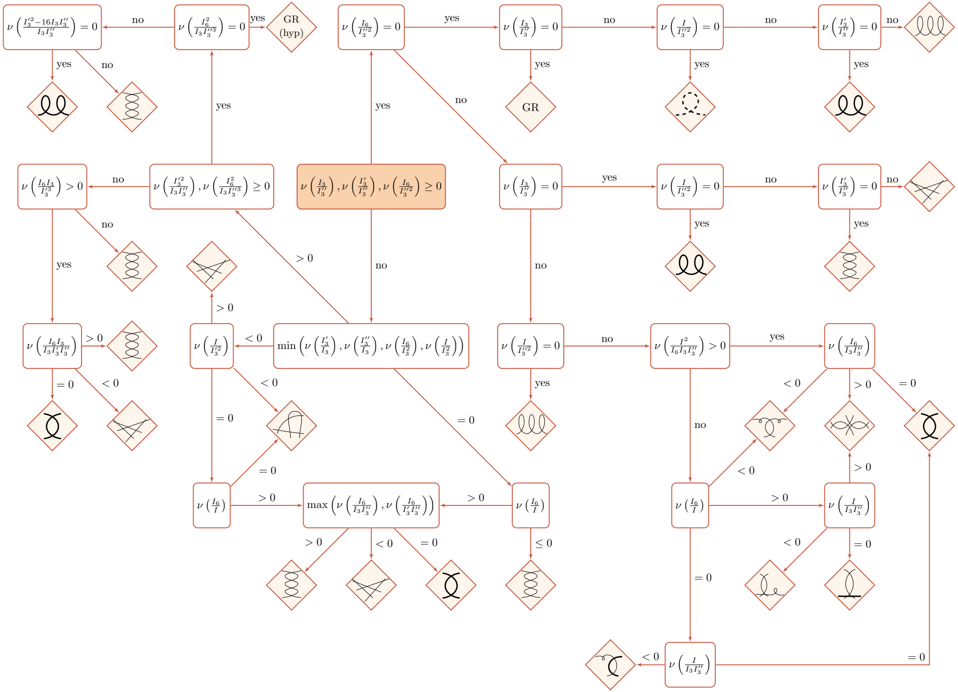


Theorem

Let Y be a Ciani curve. Then there are 13 different possibilities for the type of stable reduction of Y.

Possible stable bad reductions of a Ciani curve





$\nu\left(\frac{I_3'^2-16I_3I_3''}{I_3I_3''}\right)=0$
no
$\nu\left(\frac{I_6^2}{I_3I_3''^3}\right)=0$
yes
GR (hyp)
$\nu\left(\frac{I_6}{I_3''^2}\right)=0$
yes
$\nu\left(\frac{I_3}{I_3''}\right)=0$
no
$\nu\left(\frac{I}{I_3''^2}\right)=0$
no
$\nu\left(\frac{I_3'}{I_3''}\right)=0$
no
GR
$\nu\left(\frac{I_6I_3}{I_3'^3}\right)>0$
$\nu\left(\frac{I_3'^2}{I_3I_3''}\right),\nu\left(\frac{I_6^2}{I_3I_3''^3}\right)\geq 0$
$\nu\left(\frac{I_3}{I_3''}\right),\nu\left(\frac{I_3'}{I_3''}\right),\nu\left(\frac{I_6}{I_3''^2}\right)\geq 0$
$\nu\left(\frac{I_3}{I_3''}\right)=0$
$\nu\left(\frac{I}{I_3''^2}\right)=0$
$\nu\left(\frac{I_3'}{I_3''}\right)=0$
$\nu\left(\frac{I_6I_3}{I_3I_3'I_3''}\right)$
$\nu\left(\frac{I}{I_3'^2}\right)$
$\min\left(\nu\left(\frac{I_3'}{I_3}\right),\nu\left(\frac{I_3''}{I_3}\right),\nu\left(\frac{I_6}{I_3^2}\right),\nu\left(\frac{I}{I_3^2}\right)\right)$
$\nu\left(\frac{I}{I_3''^2}\right)=0$
$\nu\left(\frac{I^2}{I_6I_3I_3''}\right)>0$
$\nu\left(\frac{I_6}{I_3I_3''}\right)$
$\nu\left(\frac{I_6}{I}\right)$
$\max\left(\nu\left(\frac{I_6}{I_3I_3''}\right),\nu\left(\frac{I_6}{I_3'I_3''}\right)\right)$
$\nu\left(\frac{I_6}{I}\right)$
$\nu\left(\frac{I_6}{I}\right)$
$\nu\left(\frac{I}{I_3I_3''}\right)$
$\nu\left(\frac{I}{I_3I_3''}\right)$
yes
no
>0
<0
$=0$
≤ 0

Thank you for your attention!
Questions?