From Hamiltonian Mechanics to Symplectic Geometry

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Newtonian Mechanics

Nature and nature's laws lay hid in night; God said "Let Newton be" and all was light.
Alexander Pope

Newton's Law of Motion

$$m\ddot{q}(t) = F(q(t), \dot{q}(t)), \quad q(t) \in \mathbb{R}^3 \text{ point mass.}$$

Examples

Central forces: $F(q) = f(|q|)\hat{q}$.

Gravitation: $f(r) = -Cr^{-2}$.

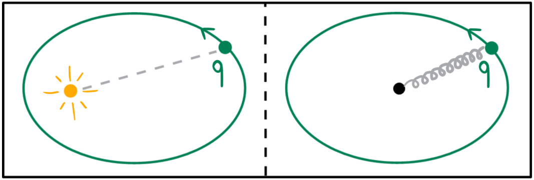
Elasticity: $f(r) = -Cr$.

"ceiiinosssttuv = Ut tensio, sic vis"

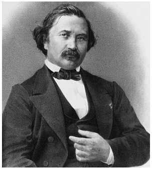
Isaac Newton (1642–1727)

Robert Hooke? (1635–1703)

Closed orbits in central forces

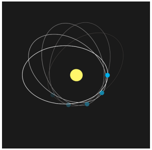


Trajectories of gravitation and elastic force



Joseph Betrand (1873):

If $F(q) = f(|q|)\hat{q}$ is an attractive central force, where all bounded trajectories are closed, then either $f(r) = -Cr^{-2}$ or $f(r) = -Cr$.



Precession of Mercury's perihelion

Attraction of other planets + General Relativity.

Conservative Systems

Lagrange (*Mécanique Analytique*, 1788)

If $F(q) = -\nabla U(q)$, then

$$m\ddot{q} = F(q, \dot{q}) \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \quad \text{(EL)}$$

where $L(q, \dot{q}) := \frac{1}{2}m|\dot{q}|^2 - U(q)$ is called Lagrangian.

Huygens' principle

$H := \frac{1}{2}m|\dot{q}|^2 + U(q)$ is conserved in time.

Christiaan Huygens (1629–1695)

Joseph-Louis Lagrange (1736–1813)

Hamilton Equations

In the space of positions $q \in \mathbb{R}^n$ and momenta $p \in \mathbb{R}^n$, the motion is given by a single function $H(q, p)$ (known as Hamiltonian):

$$\dot{q} = +\frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \qquad \text{(Ham)}$$

From Lagrange to Hamilton

- If $H = \frac{1}{2m}|p|^2 + U(q)$, then $p = m\dot{q}$ and (Ham) $\iff$ (EL).
- Hamiltonian formalism is more general than the Lagrangian one. *(If H is convex in p can define L by Legendre transformation)*
- $\dot{H} = 0$.

William Hamilton (1805–1865)

Louis Legendre (1752–1797)

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William Hamilton (1805–1865)

Adrien-Marie Legendre (1752–1833), left

Examples of Hamiltonian systems

1. Planetary motion (e.g. the 3-body problem);
2. Geodesics in curved space (e.g. on a football);
3. Charged particle in an electromagnetic potential;
4. Predator-prey models and zero-sum evolutionary games.

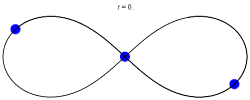
Hamiltonians 1,2,3 are of the type

$$H(q,p) = \tfrac{1}{2}|p - A(q)|_g^2 + U(q),$$

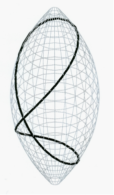
with g metric tensor, A vector potential, U scalar potential.

Hamiltonian 4 is of the type

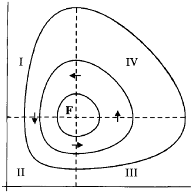
$$H(q,p) = \log q + \log p - aq - bp, \qquad a, b > 0.$$



1.



2.



4.

A geometric perspective

Klein (*Erlangen Program*, 1872)

Geometry is the study of a group of transformations (symmetries) $G = \{\varphi : X \to X\}$ on a space X.

Example

$X = \mathbb{R}^2$, $G = \{$ rigid motions $\}$ or $G' = \{$ affine transformations $\}$.

Symplectomorphisms

For us: $X = \mathbb{R}^n \times \mathbb{R}^n$, $G = \{\varphi = (Q, P) \mid \varphi$ preserves (Ham)$\}$.

Felix Klein (1849–1925)

$$\begin{cases} \dot{q} = +\dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial q}. \end{cases} \iff \begin{cases} \dot{Q} = +\dfrac{\partial H}{\partial P}, \\ \dot{P} = -\dfrac{\partial H}{\partial Q}. \end{cases}$$

Continuous symmetries

Solutions of (Ham) yield the Hamiltonian flow of H

$$\Phi_H^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad \Phi_H^t(q(0), p(0)) := (q(t), p(t)), \;\; \forall\, t.$$

- $\Phi_H^t \in G$, $\forall\, t$. ($\Rightarrow$ G is infinite-dimensional!)

Noether Theorem

$$\forall\, H_1, H_2 : \quad H_1 \circ \Phi_{H_2}^t = H_1 \;\; \forall\, t \quad \Longrightarrow \quad H_2 \circ \Phi_{H_1}^t = H_2 \;\; \forall\, t.$$

Emmy Noether (1883–1935)

"Every continuous symmetry $\Phi_{H_2}^t$ yields a conserved quantity H_2."

Example

If H is invariant under rotations around an axis $\hat{z}$, the component $p \cdot \hat{z}$ of momentum is preserved.

Enters symplectic geometry

The symplectic form is the differential two-form

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

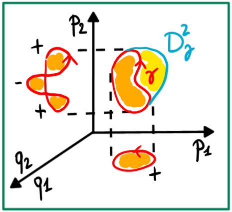
- $\varphi \in G \iff \varphi$ preserves ω.

Area measurements

If $D^2_\gamma \subset \mathbb{R}^{2n}$ is a disc bounded by a closed curve γ, then

$$A(\gamma) := \int_{D^2_\gamma} \omega$$

is the sum of the areas of the projections of D^2_γ to (q_i, p_i)-planes.

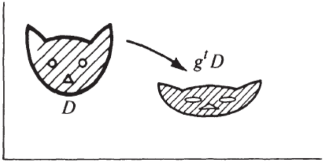


Conservation of volume and eternal return

- $\frac{1}{n!}\omega^n = \mathrm{dvol}_{2n}$ is the euclidean volume element.

$\Rightarrow$ $G \subset G' := \{F : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid F \text{ preserves volume }\}$.

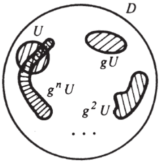
Joseph Liouville (1809–1882)



A symplectic cat [Arnold]

- If $D \subset \mathbb{R}^{2n}$ is compact and invariant for Φ_H^t, then almost every trajectory in D returns arbitrarily close to its initial condition.

Henri Poincaré (1854–1912)



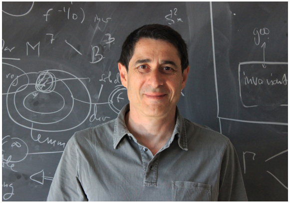
Recurrence [Arnold]

Periodic orbits

- $\Sigma := H^{-1}(h) \subset \mathbb{R}^{2n}$ compact, regular energy hypersurface. It is invariant by Φ_H. Does it contain closed orbits?
  - In general: No. (Ginzburg, 1995)
  - If $\Sigma = \partial K$ with K convex body, then Yes! (Weinstein, 1978)

Maupertuis variational principle

$\gamma \subset \Sigma$ is a closed orbit of $\Phi_H \iff \gamma$ is a stationary point for $\delta \mapsto A(\delta)$ on the space of closed loops $\delta \subset \Sigma$.



Viktor Ginzburg

Alan Weinstein

P. de Maupertuis (1698–1759)

Symplectic vs. volume-preserving

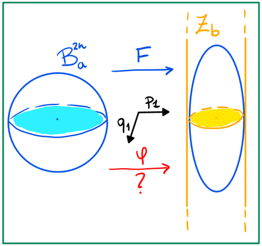
Question

How much more special is a symplectomorphism $\varphi \in G$ than a general volume-preserving map $F \in G'$ for $n > 1$?

Strategy

Take the euclidean ball $B_a^{2n} := \{(q,p) \mid \pi(|q|^2 + |p|^2) \leq a\}$.
What can $\varphi(B_a^{2n})$ be? Compare with the infinite cylinder

$$Z_b := B_b^2 \times \mathbb{R}^{2n-2} = \{\pi(q_1^2 + p_1^2) \leq b\}.$$



- For $a > b$ we can find $F \in G'$ such that $F(B_a^{2n}) \subset Z_b$.
- Can we find $\varphi \in G$ with same property?

Non-squeezing Theorem

Gromov (1985)

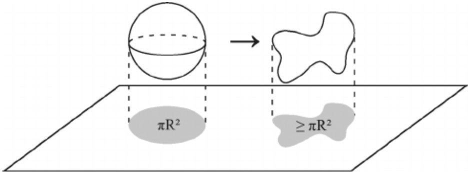
If $a > b$, then $\varphi(B_a^{2n}) \not\subset Z_b$, $\forall \varphi \in G$. Equivalently,

$$\forall\, a > 0, \quad \mathrm{vol}_2\big(\mathrm{Proj}_2[\varphi(B_a^{2n})]\big) \geq a, \quad \forall\, \varphi \in G.$$

Symplectic uncertainty principle

If we know total momentum and position up to some error, we cannot find a change of coordinates φ improving our knowledge of momentum and position in a given direction.

Mikhail Gromov (1943–)



2-dim. shadow of a symplectic ball [de Gosson]

How large is the shadow of a symplectic ball?

Let $1 < k < n$. What do we know about $\mathrm{vol}_{2k}\left(\mathrm{Proj}_{2k}[\varphi(B_a^{2n})]\right)$?

Non-linear squeezing (Abbondandolo–Matveyev, 2013)

For all $\varepsilon > 0$ there exists $\varphi_\varepsilon \in G$ with

$$\mathrm{vol}_{2k}\left(\mathrm{Proj}_{2k}[\varphi_\varepsilon(B_a^{2n})]\right) < \varepsilon.$$

Linear non-squeezing (Abbondandolo–B., 2020)

If $\varphi \in G$ is C^3-close to a linear symplectomorphism, then

$$\mathrm{vol}_{2k}\left(\mathrm{Proj}_{2k}[\varphi(B_a^{2n})]\right) \geq \frac{a^k}{k!} = \mathrm{vol}_{2k}(B_a^{2k}).$$

Alberto Abbondandolo

Embedding ellipsoids into balls

For $a \geq 1$, consider the ellipsoid $E_{1,a} \subset \mathbb{R}^4$ given by

$$E_{1,a} = \left\{\pi(|q_1|^2 + |p_1|^2) + \tfrac{\pi}{a}(|q_2|^2 + |p_2|^2) \leq 1\right\}.$$

Define $\sigma(a) := \inf\left\{b > 0 \;\middle|\; \exists\varphi \in G,\ \varphi(E_{1,a}) \subset B^4_b\right\} \geq \sqrt{a}$.

McDuff–Schlenk (2012)

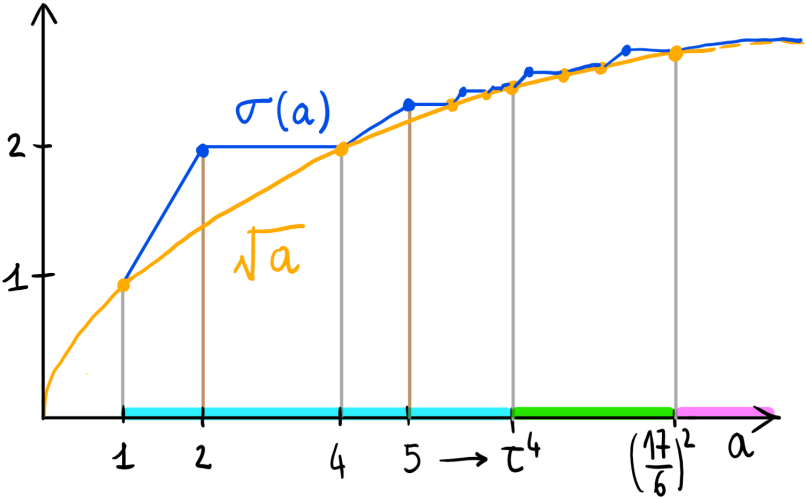
- For $(\frac{17}{6})^2 \leq a$: $\sigma(a) = \sqrt{a}$;
- For $\tau^4 \leq a \leq (\frac{17}{6})^2$: $\sigma(a)$ has transition behavior ($\tau = \frac{1+\sqrt{5}}{2}$);
- For $1 \leq a \leq \tau^4$: $\sigma(a)$ is an infinite staircase

  (inner corners at $a = \frac{F^2_{2k+3}}{F^2_{2k+1}}$ and outer corners at $a = \frac{F_{2k+5}}{F_{2k+1}}$ accumulating at $a = \tau^4$). F_k is the k-th Fibonacci number.

Dusa McDuff

Felix Schlenk

Graph of $\sigma(a)$



The graph of $\sigma(a)$ (in blue) sitting on top of the graph of $\sqrt{a}$.
On the a-axis the three different regimes.

Capacities

Definition

A symplectic capacity assigns to every subset $S \subset \mathbb{R}^{2n}$ a number $c(S) \in [0, \infty]$ such that

1. $c(S_1) \leq c(S_2)$, $\forall\, S_1 \subset S_2$;
2. $c(\varphi(S)) = c(S)$, $\forall\, S\ \forall\, \varphi \in G$;
3. $c(rS) = r^2 c(S)$, $\forall\, r > 0$;
4. $c(B_a^{2n}) = a = c(Z_a)$.

Remark

- By 1. and 2. capacities obstruct symplectic embeddings.
- Gromov's non-squeezing $\iff$ Existence of a capacity.
- Consider ellipsoid $E_{a_1,\dots,a_n} \subset \mathbb{R}^{2n}$ with $a_1 \leq \ldots \leq a_n$.
  - Then: $B^{2n}(a_1) \subset E_{a_1,\dots,a_n} \subset Z_{a_1}$.
  - Thus: $c(E_{a_1,\dots,a_n}) = a_1$ for all capacities c.
- If $0 < c(B_a^{2n}) \leq c(Z_a) < +\infty$ instead of 4., then c is called generalized symplectic capacity.

Capacity against volume

How much do $c(S)$ and $\mathrm{vol}_{2n}(S)$ differ for $S \subset \mathbb{R}^{2n}$?

Definition: The c-systolic ratio of $S \subset \mathbb{R}^{2n}$ is defined by

$$\rho_c(S) := \frac{c(S)^n}{\mathrm{vol}_{2n}(S)}.$$

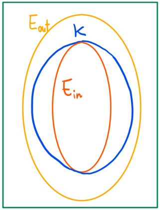
Note: $\rho_c(E_{a_1,\dots,a_n}) \leq n!$. Equality $\iff$ $E_{a_1,\dots,a_n} = B^{2n}(a)$.

Viterbo conjecture (2000): If $K \subset \mathbb{R}^{2n}$ is a convex body, then

$$\rho_c(K) \leq n! \qquad \left[\rho_c(K) = n! \iff \varphi(K) = B^{2n}(a)\right].$$

Artstein-Avidan–Milman–Ostrover (2008): $\exists C > 0$, $\rho_c(K) \leq Cn!$.

Claude Viterbo



Ellipsoids approximating K

Capacities from periodic orbits

Ekeland–Hofer–Zehnder (80s)

There exists a capacity c_{EHZ} such that for all convex bodies K:

$$c_{EHZ}(K) := \min \left\{ A(\gamma) \,\middle|\, \gamma \text{ closed orbit of } (\mathrm{Ham}) \text{ on } \partial K = H^{-1}(h) \right\}.$$

Question: Does the Viterbo conjecture hold for $c = c_{EHZ}$?

Local systolic inequality (Abbondandolo–B., 2020)

Viterbo conjecture is true if $c = c_{EHZ}$ and K is C^3-close to $B^{2n}(a)$.

Ivar Ekeland

Helmut Hofer

Eduard Zehnder

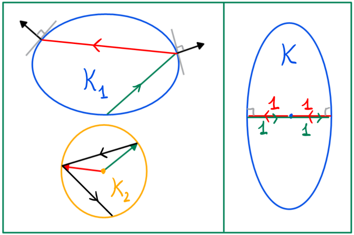
Playing billiards

Take $K = \mathcal{K}_1 \times \mathcal{K}_2$ with $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^n$ centrally symmetric convex bodies ($\mathcal{K}_i$ is the unit ball of a norm $\|\cdot\|_{\mathcal{K}_i}$).

- The Hamiltonian flow on ∂K is obtained by playing billiard in the table $\mathcal{K}_1$ using the reflection law of $\mathcal{K}_2$.
- $c_{EHZ}(\mathcal{K}_1 \times \mathcal{K}_2) =$ "$\|\cdot\|_{\mathcal{K}_2}$-length of the shortest closed billiard trajectory in $\mathcal{K}_1$".

Artstein-Avidan–Karasev–Ostrover (2014): $c_{EHZ}(\mathcal{K} \times \mathcal{K}^o) = 4$, where $\mathcal{K}^o$ is the polar dual of $\mathcal{K}$. As a consequence:

$$\rho_{EHZ}(\mathcal{K} \times \mathcal{K}^o) = \frac{4^n}{\mathrm{vol}_n(\mathcal{K})\mathrm{vol}_n(\mathcal{K}^o)}.$$



Billiard game and orbit of length 4.

How round is a convex body?

Let $\mathcal{K} \subset \mathbb{R}^n$ centrally symmetric convex body. Can we measure its pointness or roundness? Let $M(\mathcal{K}) := \mathrm{vol}_n(\mathcal{K})\mathrm{vol}_n(\mathcal{K}^o)$.

Mahler conjecture (1939):

$$\frac{4^n}{n!} = M([0,1]^n) \leq M(\mathcal{K}) \leq M(B^n).$$

- Blaschke–Santaló (1917, 1949): $M(\mathcal{K}) \leq M(B^n)$ holds. Equality only for $\mathcal{K} = B^n$.
- $M([0,1]^n) \leq M(\mathcal{K})$ is still open. Viterbo $\Rightarrow$ Mahler. Proven for $n = 2$ (Mahler), for $n = 3$ (Iriyeh–Shibata, 2020).

Kurt Mahler (1903–1988)