Hyperbolic geometry and cluster algebras

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The hyperbolic plane

Hyperbolic geometry was discovered and developed 150–200 years ago by many scientist (Beltrami, Bolyai, Gauss, Lobachevsky, Klein, Taurinus...), after around two thousend years of attempts to deduce Euclid's fifth axiom from the first four.

Some properties of the hyperbolic plane are:

1. Given a hyperbolic geodesic and a point outside of it, there are infinitely many geodesics that pass through the latter and are parallel to the former;
2. it has several models: the Poincaré disc $\mathbb{D}$, the upper half plane $\mathbb{U}$, the hyperboloid, the Beltrami-Klein disc;
3. for $\mathbb{D}$ y $\mathbb{U}$:
   - $\mathbb{D} \subseteq \mathbb{C}$, $\mathbb{U} \subseteq \mathbb{C}$;
   - the unit circle $\mathbb{S}^1$ functions as a set of points at infinity for $\mathbb{D}$, whereas $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ plays such role for $\mathbb{U}$;
   - the hyperbolic geodesics are the segments of Euclidean circles that are perpendicular to $\mathbb{S}^1$ or $\overline{\mathbb{R}}$, respectively;
   - the hyperbolic circles are precisely the Euclidean circles that are fully contained in $\mathbb{D}$ or $\mathbb{U}$, respectively.

Theorem

For $\mathbb{D}$ *and* $\mathbb{U}$, *the group of orientation-preserving isometries is precisely the group of Riemann surface automorphisms of* $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ *that preserve* $\mathbb{D}$ *and* $\mathbb{U}$, *respectively. That is,*

$SU_{1,1} = \{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \}$

$$\mathrm{Iso}^+(\mathbb{D}) = \{\nu \in \mathrm{Mob}(\overline{\mathbb{C}}) \mid \nu(\mathbb{D}) = \mathbb{D}\} = \mathrm{PSU}_{1,1}$$
$$\mathrm{Iso}^+(\mathbb{U}) = \{\nu \in \mathrm{Mob}(\overline{\mathbb{C}}) \mid \nu(\mathbb{U}) = \mathbb{U}\} = \mathrm{PSL}_2(\mathbb{R})$$

$SL_2(\mathbb{R}) = \{ A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1 \}$

Theorem

Given two ordered triples of distinct points of $\overline{\mathbb{C}}$, *say* (z_1, z_2, z_3) *and* (w_1, w_2, w_3), *there exists exactly one Möbius transformation* $\nu \in \mathrm{Mob}(\overline{\mathbb{C}})$ *such that*

$$\begin{aligned} \nu(z_1) &= w_1, \\ \nu(z_2) &= w_2 \\ \text{and} \quad \nu(z_3) &= w_3. \end{aligned}$$

Teichmüller space

Definition

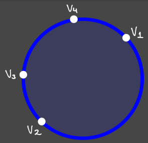
An ideal polygon in $\mathbb{D}$ or $\mathbb{U}$ is an h-convex polygon whose vertices are points at infinity.

Observation

Drawing an ideal polygon with n vertices/sides amounts to choosing a set of n points from $\mathbb{S}^1$ or $\overline{\mathbb{R}}$.

Definition

An ideal polygon with well-ordered vertices is an ordered tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of points that appear in $\mathbf{v}$ in the clockwise sense of $\mathbb{S}^1$.



Observation

Any given ideal polygon P with n vertices/sides underlies exactly n ideal polygons with well-ordered vertices, one just has to choose a vertex of P and designate it as the first vertex.

Question

Take an Euclidean polygon C_n with n vertices $w_1, \ldots, w_n$, ordered in the clockwise sense. How many esencially distinct hyperbolic metrics can we impose on C_n that make it an ideal polygon (with well-ordered vertices)?

Definition

The Teichmüller space of (C_n, w_1) is

$$\mathcal{T}(C_n, w_1) := \{\mathbf{v} \mid \mathbf{v} \textit{ is an ideal polygon with } n \textit{ well-ordered vertices}\} / \operatorname{Iso}^+(\mathbb{H})$$

Theorem

$$\mathcal{T}(C_n, w_1) \quad \cong \quad \mathbb{R}_{>0}^{n-3}$$

Definition

The decorated Teichmüller space of (C_n, w_1) is

$$\widetilde{\mathcal{T}}(C_n, w_1) := \{(\mathbf{v}, \mathbf{h}) \mid \mathbf{v} = (v_1, \dots, v_n) \text{ is an ideal polygon with } n \text{ well-ordered vertices, } \mathbf{h} = (h_1, \dots, h_n) \text{ is an } n\text{-tuple of horocycles, with } h_j \text{ based at } v_j\} / \operatorname{Iso}^+(\mathbb{H})$$

Theorem

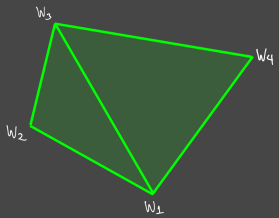
$$\widetilde{\mathcal{T}}(C_n, w_1) \cong \mathbb{R}^{2n-3}_{>0}$$

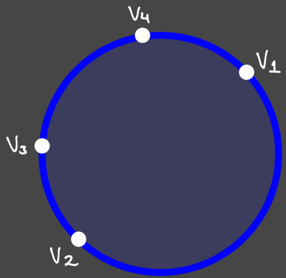
Question

How to paremeterize $\widetilde{\mathcal{T}}(C_n, w_1)$ in such a way that the $2n-3$ parameters have the same nature?

Observation

Fix a combinatorial diagonal (j, k)*,* $j < k$*, of* (C_n, w_1)*. For each* $\mathbf{v} \in \mathcal{T}(C_n, w_1)$*,* (j, k) *induces a hyperbolic geodesic* $[v_j, v_k]_{\mathbb{H}}$ *connecting* v_j *and* v_k*.*



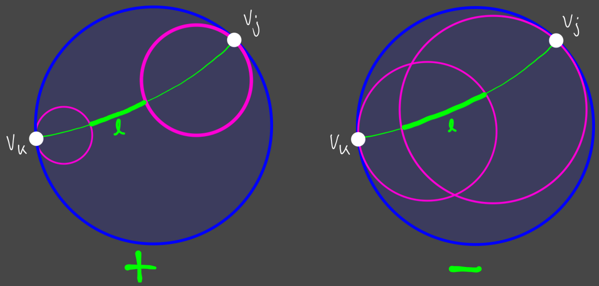


Definition (Penner ~2004)

Given any combinatorial diagonal (j,k), $j<k$, *of* C_n, *the* *lambda length* *of* $(\mathbf{v},\mathbf{h}) = ((v_1,\dots,v_n),(h_1,\dots,h_n))$ *with respect to* (j,k) *as*

$$\lambda_{(j,k)}(\mathbf{v},\mathbf{h}) := \sqrt{e^{\pm\ell}},$$

where:



Thus, for each diagonal (j, k), $j < k$, of C_n, we have a function

$$\lambda_{(j,k)} : \widetilde{\mathcal{T}}(C_n, w_1) \to \mathbb{R}_{>0} \subseteq \mathbb{R}$$

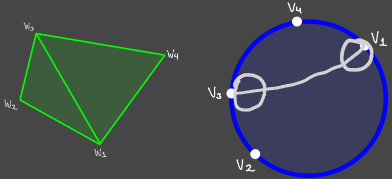
Theorem (Penner ~2004)

For any given combinatorial triangulation T of C_n (including in T the n boundary segments), the lambda lengths with respect to the diagonals belonging to T yield a bijection (actually, a diffeomorphism)

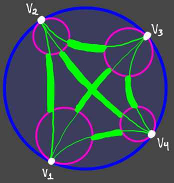
$$\lambda_T : \widetilde{\mathcal{T}}(C_n, w_1) \to \mathbb{R}^{2n-3}_{>0}$$

$$(\mathbf{v}, \mathbf{h}) \mapsto (\lambda_{(j,k)}(\mathbf{v}, \mathbf{h}))_{(j,k)\in T}.$$

Example

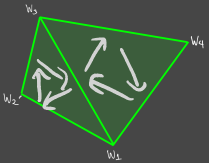


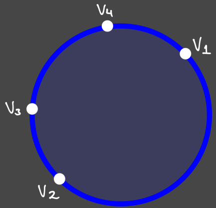
Theorem (Penner ~2004)



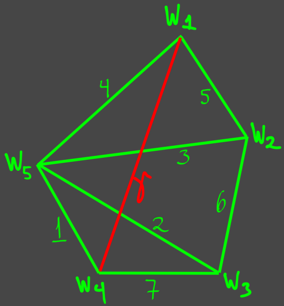
$$\lambda_{(1,3)}\lambda_{(2,4)} = \lambda_{(1,2)}\lambda_{(3,4)} + \lambda_{(1,4)}\lambda_{(2,3)}$$

Example

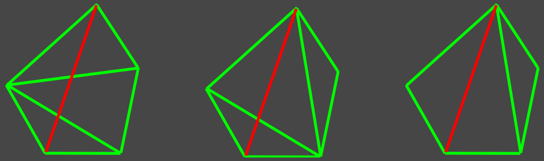




$$\lambda_{(2,4)} = \frac{\lambda_{(1,2)}\lambda_{(3,4)}+\lambda_{(1,4)}\lambda_{(2,3)}}{\lambda_{(1,3)}}$$

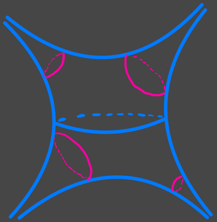


$$\lambda_\gamma = \frac{\lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_4\lambda_6 + \lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3}$$



Theorem (Fock-Goncharov, Fomin-Shapiro-Thurston, Fomin-Thurston, Gekhtman-Shapiro-Vainshtein, Penner)

All of the above remains true if instead of (C_n, w_1) one takes a surface with marked points.



Cluster algebras

The starting point is a pair $(Q, \mathbf{x})$ consisting of a 2-acyclic quiver Q with vertices $1, \ldots, n$, and an n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ algebraically independent over the ground field F. Any such pair is referred to as a seed, $\mathbf{x}$ is the cluster of the seed, the elements $x_1, \ldots, x_n$ are the cluster variables of the seed.

Definition (Sergey Fomin, Andrei Zelevinsky, ~2002)

Given a seed $(Q, \mathbf{x})$ and a vertex k of Q, the mutation of $(Q, \mathbf{x})$ with respect to k is the seed $\mu_k(Q, \mathbf{x}) := (\mu_k(Q), \mu_k(\mathbf{x}))$, where:

(Quiver mutation in 3 steps)

1. *for each pair $j \to k \to i$ in Q, add a a new arrow $j \to i$;*
2. *reverse the arrows incident to k;*
3. *remove oriented 2-cycles.*

Result $=:$ $\mu_k(Q)$

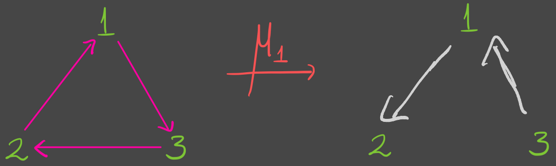
(Cluster mutation)

$$\mu_k(\mathbf{x}) := (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$$

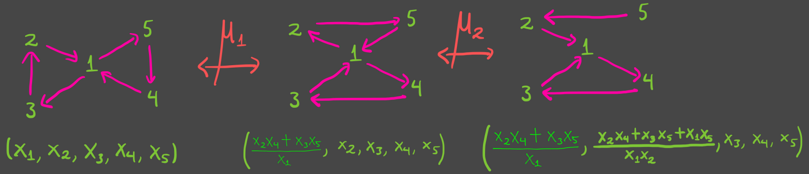
$$x'_k := \frac{\prod_{j \to k} x_j + \prod_{k \to i} x_i}{x_k}$$

Example (Quiver mutation)

$Gr_2(\mathbb{C}^n)$



Example (Seed mutation (:= quiver mutation + cluster mutation))



Definition (Fomin-Zelevinsky ~2002)

The cluster algebra associate to the seed $(Q, \mathbf{x})$ is the F-algebra generated by all the cluster variables that appear in the seeds obtained by applying arbitrary mutation sequences to $(Q, \mathbf{x})$.

Theorem (Penner ~2004, Fomin-Thurston 2008–2012)

The lambda length coordinate ring of the decorated Teichmüller space $\widetilde{\mathcal{T}}(C_n, w_1)$ is a cluster algebra over $F = \mathbb{R}$.

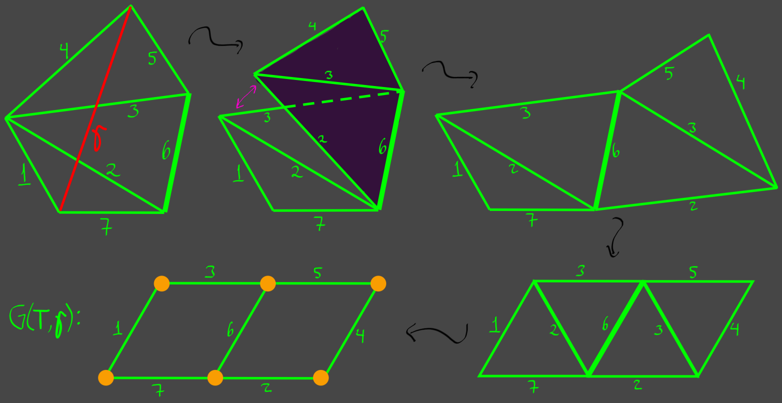
$$\widetilde{T}(C_n, w_1) \longrightarrow \mathbb{R}$$

Theorem (Fomin-Thurston, 2008–2012)

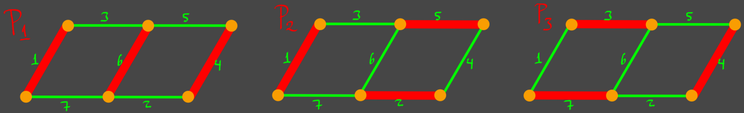
More generally, for any surface with marked points, the lambda length coordinate ring of its decorated Teichmüller space is a cluster algebra over $F = \mathbb{R}$.

Bipartite graphs and perfect matchings

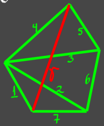
Musiker-Schiffler-Williams: *reverse origami* $\rightsquigarrow$ bipartite graph $G(\tau,\gamma)$.



Perfect matchings of $G(\tau, j)$:



$\lambda(P_1) := \lambda_1 \lambda_4 \lambda_6$ $\lambda(P_2) := \lambda_1 \lambda_2 \lambda_5$ $\lambda(P_3) := \lambda_3 \lambda_4 \lambda_7$

Recalling that  , consider the quotient

$$\frac{\sum_P \lambda(P)}{\text{mono}(\tau, j)} := \frac{\lambda_1\lambda_4\lambda_6 + \lambda_1\lambda_2\lambda_5 + \lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3},$$

it is precisely $\lambda_\gamma = \dfrac{\lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_4\lambda_6 + \lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3}$!!!

Theorem (Musiker-Schiffler-Williams, ~2011)

For any combinatorial triangulation T or (C_n, w_1) and any diagonal $\gamma \notin T$:

$$\lambda_\gamma = \frac{\sum_P \lambda(P)}{\mathrm{mono}(T,\gamma)}$$

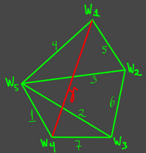
where the sum runs over all perfect matchings P of $G(T,\gamma)$.

Representations of quivers

Definition

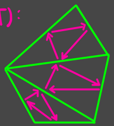
A representation M of a quiver Q assigns a $\mathbb{C}$-vector space M_j to each vertex j, and a $\mathbb{C}$-linear transformation $M_a : M_j \to M_k$ to each arrow $a : j \to k$.

Each triangulation T has an associated quiver $Q(T)$. It is possible to associate to γ a representation $M(T, \gamma)$ of $Q(T)$.

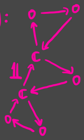


$$\lambda_\gamma = \frac{\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_4\lambda_6+\lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3}$$

Q(T):



M(T,γ):



Thank you!