Coupled Oscillator Networks: Structure, Interactions, and Dynamics

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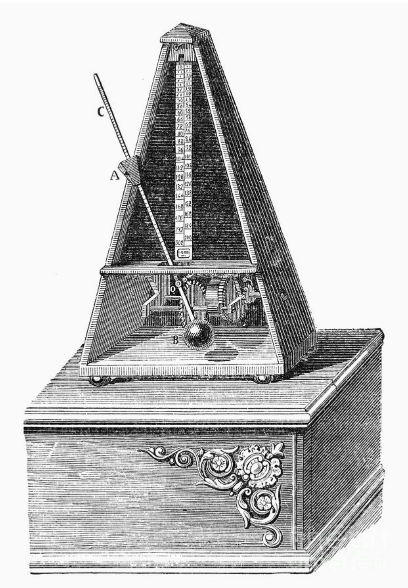
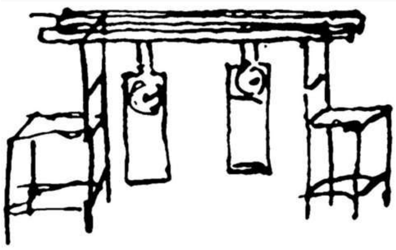
... but thanks also go to various other students and collaborators.

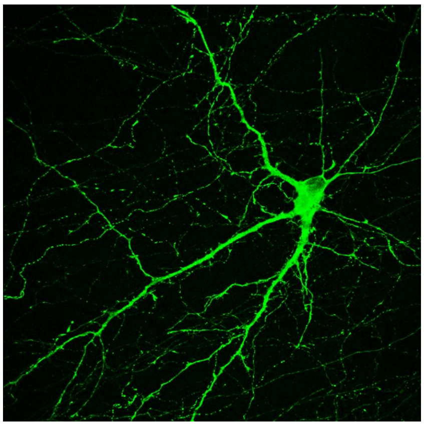
Coupled Oscillator Networks

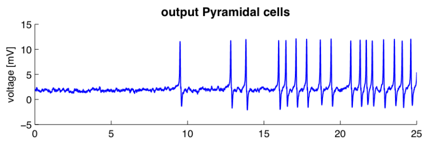
Clocks



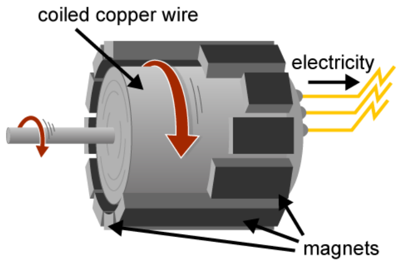


Brain





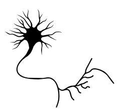
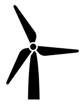
Power grids

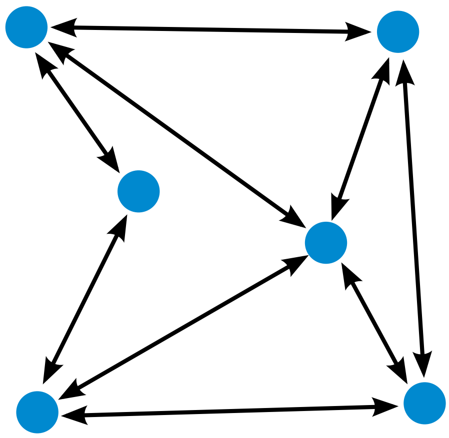


Coupled Metronomes are a Network Dynamical Systems

https://www.youtube.com/watch?v=T58lGKREubo

Network Dynamical Systems

Network of oscillatory units: ● $\in$ { , , ... }



Network structure (‘topology’):
Who interacts with whom?

Network interaction:
How does one oscillator influence the other?

Network dynamics:
Collective dynamics of all nodes.

Q: How do **structure** and **interactions** shape the **network dynamics**?

From Oscillators to Phase Oscillators

Weakly coupled **nonlinear oscillators** with state $x_k \in \mathbb{R}^Q$

$$\dot{x}_k = F_k(x_k) + \varepsilon \sum_{j=1}^{N} G_{kj}(x_j, x_k).$$

$\downarrow$

Phase reduction, phase response curve (PRC) $Z(\phi)$, interactions $h_{kj}(t)$

$\downarrow$

Average over fast oscillations

$\downarrow$

(Averaged) **Phase oscillator network** with state $\theta_k \in \mathrm{T} = \mathbb{R}/2\pi\mathbb{Z}$

$$\dot{\theta}_k = \omega_k + \sum_{j=1}^{N} g_{kj}(\theta_j - \theta_k)$$

Interaction: **Coupling functions** g_{kj}. *Kuramoto model* with $g_{kj} = \sin$.

Ashwin, Coombes, Nicks (2016). J Math Neuro, 6(1), 2.

Modular Networks

1. Population
2. Populations
3. Populations

Phase Reduction in Action

Consider N symmetric oscillators with $z_k \in \mathbb{C}$ close to a Hopf bifurcation

$$\dot{z}_k = F_\lambda(z_k) + \varepsilon G_\lambda(z_k; z_1, \ldots, z_N).$$

Phase Reduction in Action

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$$\dot{z}_k = F_\lambda(z_k) + \varepsilon G_\lambda(z_k; z_1, \ldots, z_N).$$

Phase approximation, $\theta_k \in \mathrm{T} = \mathbb{R}/2\pi\mathbb{Z}$, valid for $t = O(\varepsilon^{-1}\lambda^{-1})$ is

$$\dot{\theta}_k = \tilde{\omega}(\theta, \varepsilon) + \frac{\varepsilon}{N}\sum_{j=1}^{N} g_2(\theta_j - \theta_k)$$

where $\tilde{\omega}(\theta, \varepsilon)$ is a S_N-symmetric function in the phases and

$$g_2(\phi) = \xi_1^0 \cos(\phi + \chi_1^0)$$

P Ashwin and A Rodrigues, Physica D 32:14–24, 2016

Phase Reduction in Action

Consider N symmetric oscillators with $z_k \in \mathbb{C}$ close to a Hopf bifurcation

$$\dot{z}_k = F_\lambda(z_k) + \varepsilon G_\lambda(z_k; z_1, \dots, z_N).$$

Phase approximation, $\theta_k \in \mathrm{T} = \mathbb{R}/2\pi\mathbb{Z}$, valid for $t = O(\varepsilon^{-1}\lambda^{-2})$ is

$$\begin{aligned}\dot{\theta}_k = \tilde{\omega}(\theta, \varepsilon) &+ \frac{\varepsilon}{N}\sum_{j=1}^{N} g_2(\theta_j - \theta_k) + \frac{\varepsilon}{N^2}\sum_{j,l=1}^{N} g_3(\theta_j + \theta_l - 2\theta_k) \\ &+ \frac{\varepsilon}{N^2}\sum_{j,l=1}^{N} g_4(2\theta_j - \theta_l - \theta_k) + \frac{\varepsilon}{N^3}\sum_{j,l,m=1}^{N} g_5(\theta_j + \theta_l - \theta_m - \theta_k)\end{aligned}$$

where $\tilde{\omega}(\theta, \varepsilon)$ is a S_N-symmetric function in the phases and

$$\begin{aligned} g_2(\phi) &= \xi_1^0 \cos(\phi + \chi_1^0) + \lambda\xi_1^1 \cos(\phi + \chi_1^1) + \lambda\xi_2^1 \cos(2\phi + \chi_2^1), \\ g_3(\phi) &= \lambda\xi_3^1 \cos(\phi + \chi_3^1), \quad g_4(\phi) = \lambda\xi_4^1 \cos(\phi + \chi_4^1), \\ g_5(\phi) &= \lambda\xi_5^1 \cos(\phi + \chi_5^1). \end{aligned}$$

P Ashwin and A Rodrigues, Physica D 32:14–24, 2016

What is Important?

First-Order Approximation

$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} \sin(\theta_j - \theta_k + \alpha)$$

Kuramoto–Sakaguchi equations: Integrable with 2 degrees of freedom.

Second-Order Approximation

$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} \sin(\theta_j - \theta_k + \alpha) + \cdots + \sum_{j,l=1}^{N} \sin(\theta_j + \theta_l - 2\theta_k + \hat{\alpha}) + \cdots$$

Network dynamical system with *higher-order interactions*.

F Battiston *et al* (2020). Physics Reports, 874, 1–92.

What is Important?

First-Order Approximation

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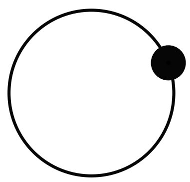
Network dynamical system with *higher-order interactions*.

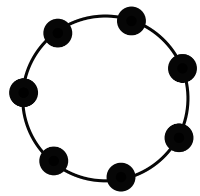
Advertisement! Review: CB, H. Harrington, E. Gross, and M. Schaub. *What are higher-order networks?* To go to SIAM Review (any day).

F Battiston *et al* (2020). Physics Reports, 874, 1–92.

Symmetric Consequences

Invariant phase configurations

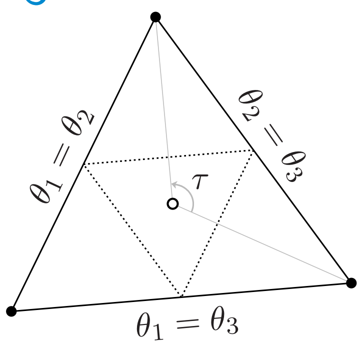




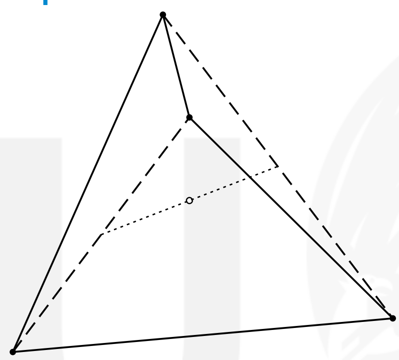
$$\mathrm{S} = \{\theta_1 = \cdots = \theta_N\} = \bullet \qquad \mathrm{D} = \{\theta_{k+1} = \theta_k + 2\pi/N\} = \circ$$

Phase ordering is preserved

$N = 3$



$N = 4$



P Ashwin, CB, and O Burylko, Front. Appl. Math. Stat. 2(7), 1–16, 2016.

Phase Oscillators with Higher-Order Interactions

Q: Are there chaotic attractors with nonpairwise coupling?

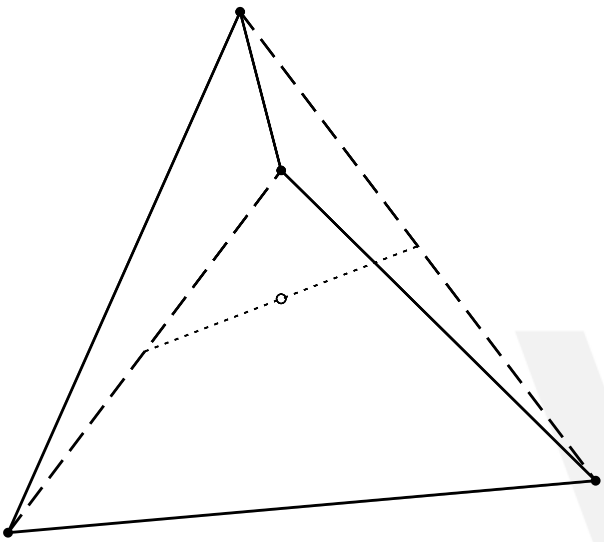
CB, P Ashwin, and A Rodrigues (2016). Chaos, 26(9), 94814.

Phase Oscillators with Higher-Order Interactions

Q: Are there chaotic attractors with nonpairwise coupling?

Fix $N = 4$, $\lambda = 1$, $\xi = (-0.3, 0.3, 0.02, 0.8, 0.02)$ and parametrize

$$g_2(\phi) = \xi_1 \cos(\phi + \chi_1) + \xi_2 \cos(2\phi + \chi_2), \quad g_3(\phi) = \xi_3 \cos(\phi + \chi_3),$$
$$g_4(\phi) = \xi_4 \cos(\phi + \chi_4), \quad g_5(\phi) = \xi_5 \cos(\phi + \chi_5).$$



CB, P Ashwin, and A Rodrigues (2016). Chaos, 26(9), 94814.

Phase Oscillators with Higher-Order Interactions

A: There chaotic attractors with nonpairwise coupling!

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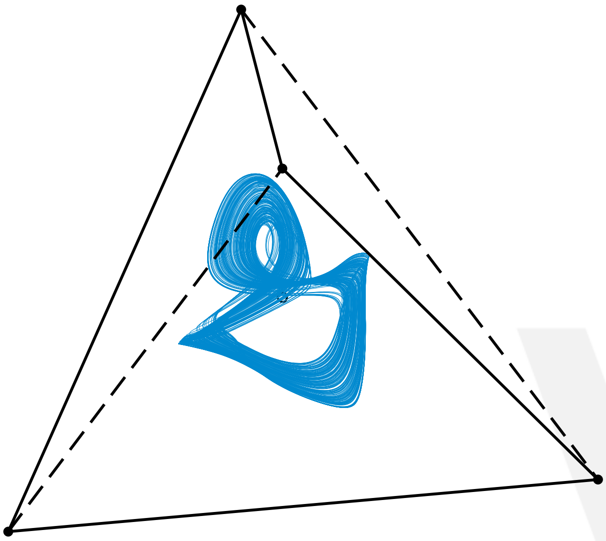

Parameters
$\chi = (0.108, 0.27, 0, 1.5, 0)$.

CB, P Ashwin, and A Rodrigues (2016). Chaos, 26(9), 94814.

Phase Oscillators with Higher-Order Interactions

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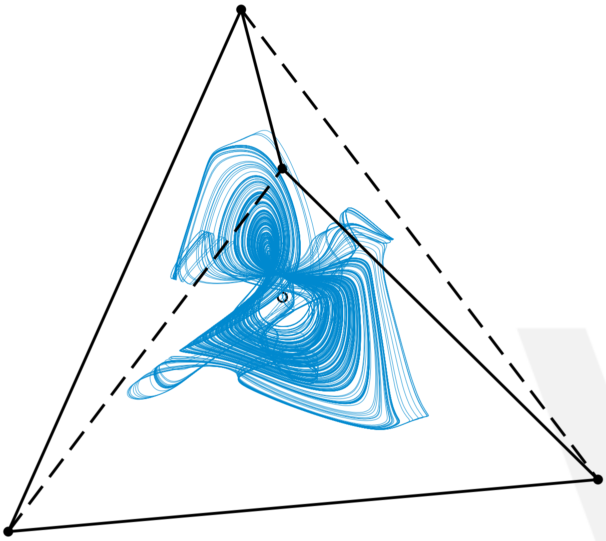

Parameters
$\chi = (0.154, 0.318, 0, 1.74, 0)$.

CB, P Ashwin, and A Rodrigues (2016). Chaos, 26(9), 94814.

Phase Oscillators with Higher-Order Interactions

A: There chaotic attractors with nonpairwise coupling!

Fix $N = 4$, $\lambda = 1$, $\xi = (-0.3, 0.3, 0.02, 0.8, 0.02)$ and parametrize

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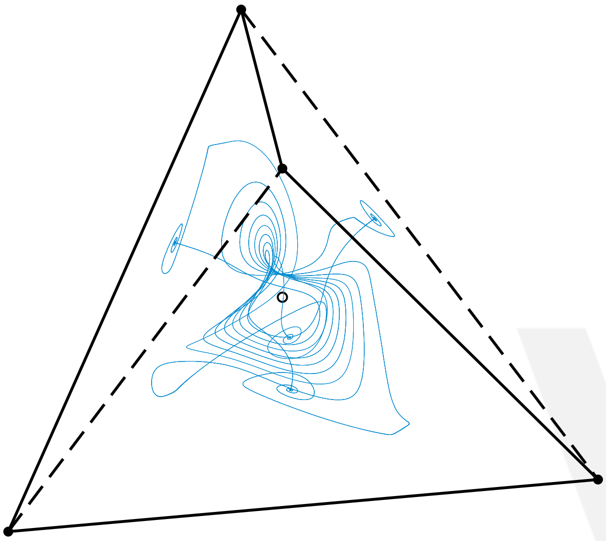


Parameters
$\chi = (0.2, 0.316, 0, 1.73, 0)$.

CB, P Ashwin, and A Rodrigues (2016). Chaos, 26(9), 94814.

Frequencies

Instantaneous frequency

$$\dot{\theta}_k(t)$$

Asymptotic average frequency

$$\Omega_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{\theta}_k(t)\,\mathrm{d}t$$

Identical all-to-all coupling, S_N symmetry: for all k, j

$$\Omega_k = \Omega_j.$$

No frequency separation for identical (1st order) phase oscillators.

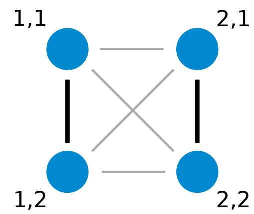
Modular Networks

1. Population
2. Populations
3. Populations

Coupled Phase Oscillator Populations

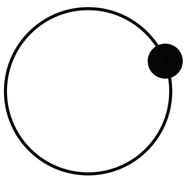
Oscillator k in population σ has phase

$$\theta_{\sigma,k} \in \mathsf{T}.$$

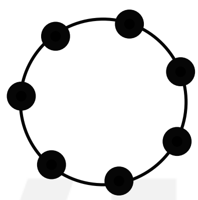


Identical oscillators: can exchange any two oscillators while preserving the equations of motion. Have $\omega_{\sigma,k} = \omega$.

Phase configurations



$$\mathrm{S} = \{\theta_{\sigma,1} = \cdots = \theta_{\sigma,N}\}$$



$$\mathrm{D} = \{\theta_{\sigma,k+1} = \theta_{\sigma,k} + 2\pi/N\}$$

Write

$$\mathrm{S}, \mathrm{D} \qquad \mathrm{SS}, \mathrm{SD}, \mathrm{DS}, \mathrm{DD} \qquad \mathrm{SSS}, \mathrm{SSD}, \mathrm{SDD}, \ldots$$

Frequencies

Frequencies

$$\dot{\theta}_{\sigma,k}(t) \qquad\qquad \Omega_{\sigma,k} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \dot{\theta}_{\sigma,k}(t)\,\mathrm{d}t$$

Frequency synchrony in population σ: for $k \neq j$

$$\Omega_{\sigma,k} = \Omega_{\sigma,j}$$

Weak chimera characterized by **localized frequency synchrony**

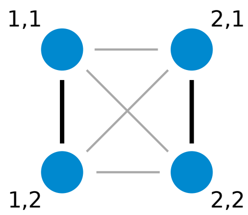
$$\begin{aligned} \Omega_{\sigma,k} &= \Omega_{\sigma,j} && \text{for any } \sigma \text{ and } j \neq k \\ \Omega_{\sigma,k} &\neq \Omega_{\tau,k} && \text{for } \sigma \neq \tau. \end{aligned}$$

Dynamics of identical oscillators show distinct frequencies.

Dynamics with Localized Frequency Synchrony

Ashwin and Burylko: There is localized frequency synchrony for weakly (globally and identically) coupled populations.

1. **Two uncoupled populations**: SD is invariant.
2. SD has **localized frequency synchrony**.
3. **Persistence** for small $\varepsilon > 0$.



P Ashwin and O Burylko (2015). Chaos, 25:13106.

Dynamics with Localized Frequency Synchrony

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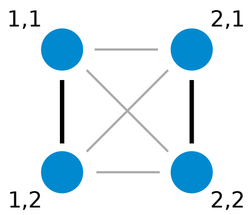
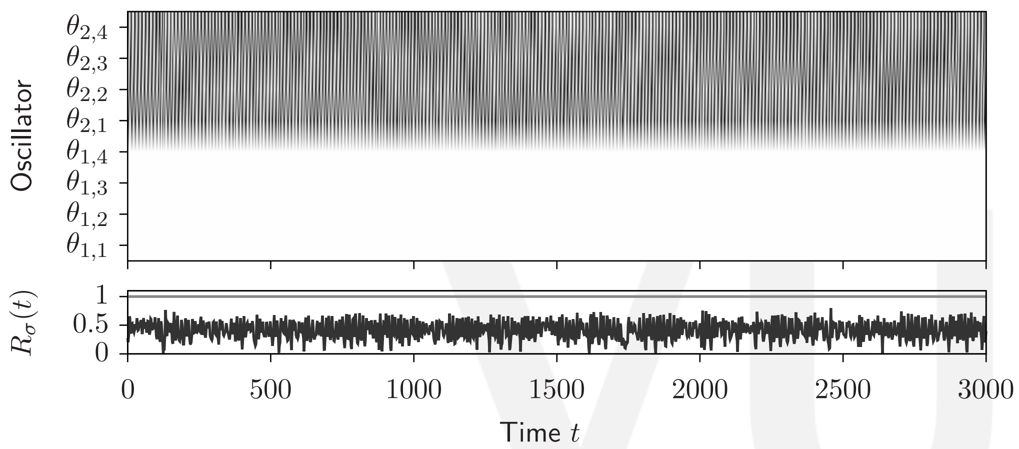


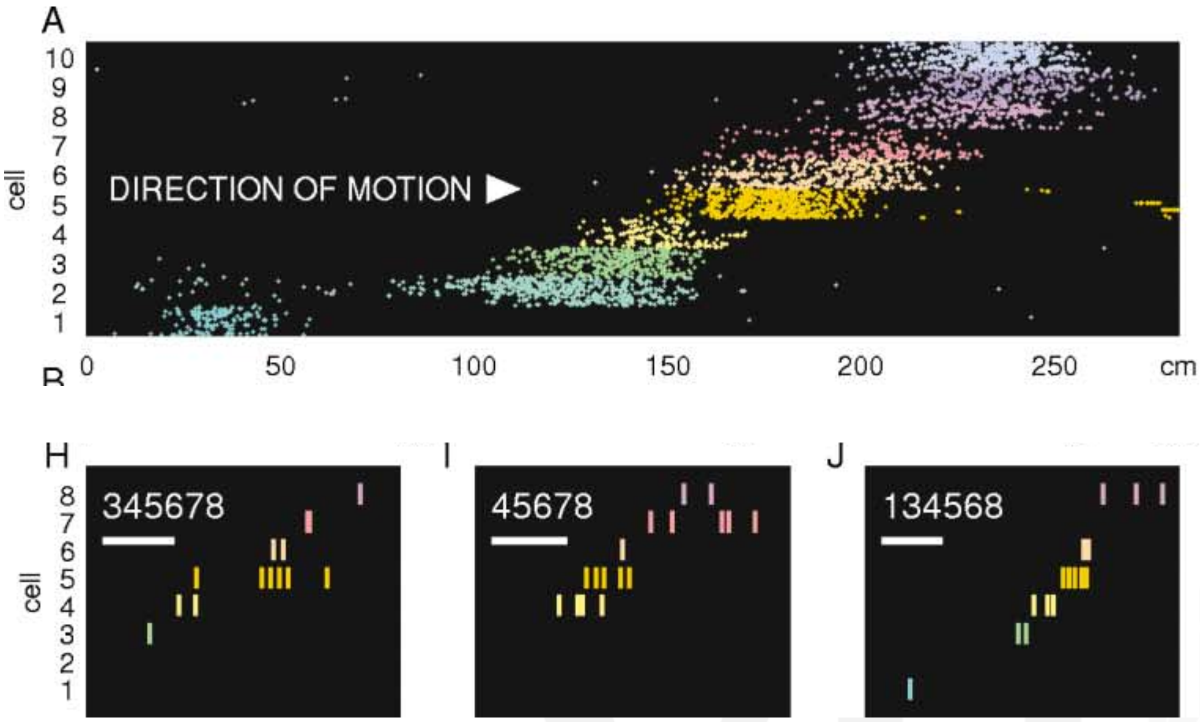
B and Ashwin: Generalization and larger populations.



P Ashwin and O Burylko (2015). Chaos, 25:13106. **CB and P Ashwin (2016). Nonlinearity, 29(5), 1468–1486.**

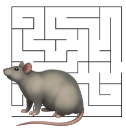
Modular Networks

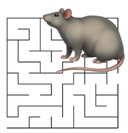
Neuroscience

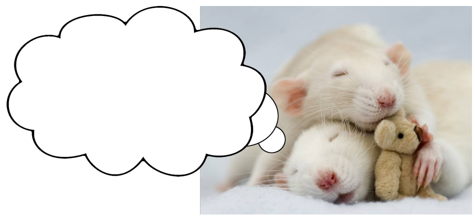


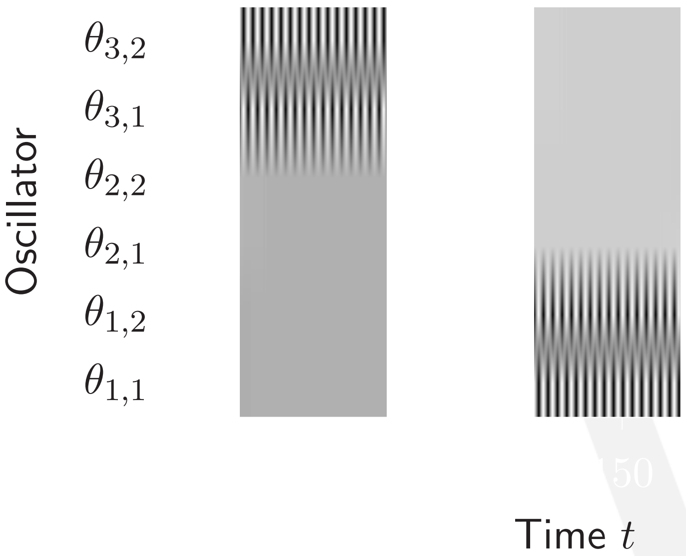
Lee and Wilson (2002). Neuron, 36(6), 1183–1194.

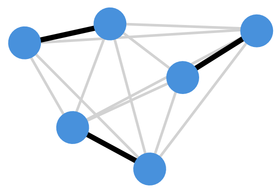
"Neuroscience"







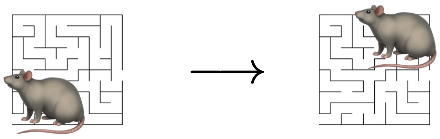


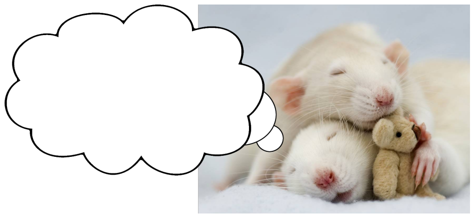


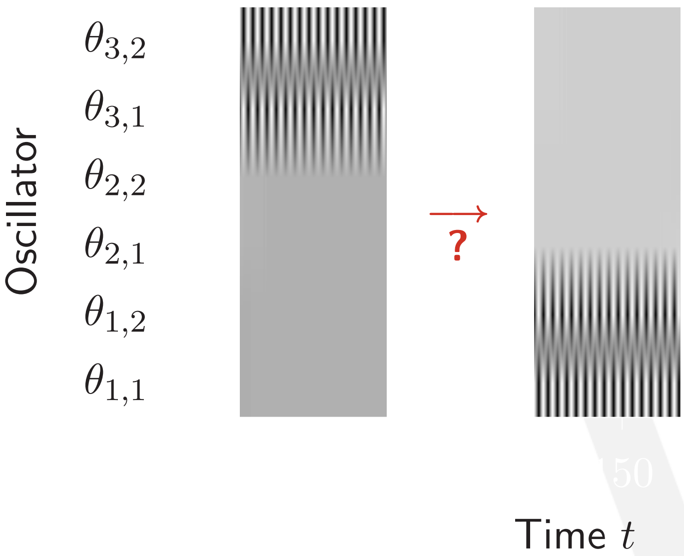
State of ● :

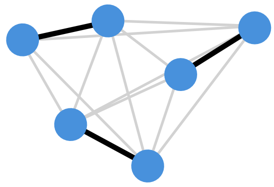
$\theta_{\sigma,k} \in [0, 2\pi)$

"Neuroscience"







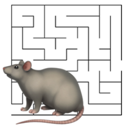
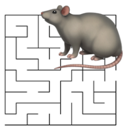


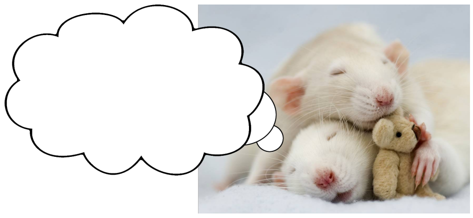
State of ● :

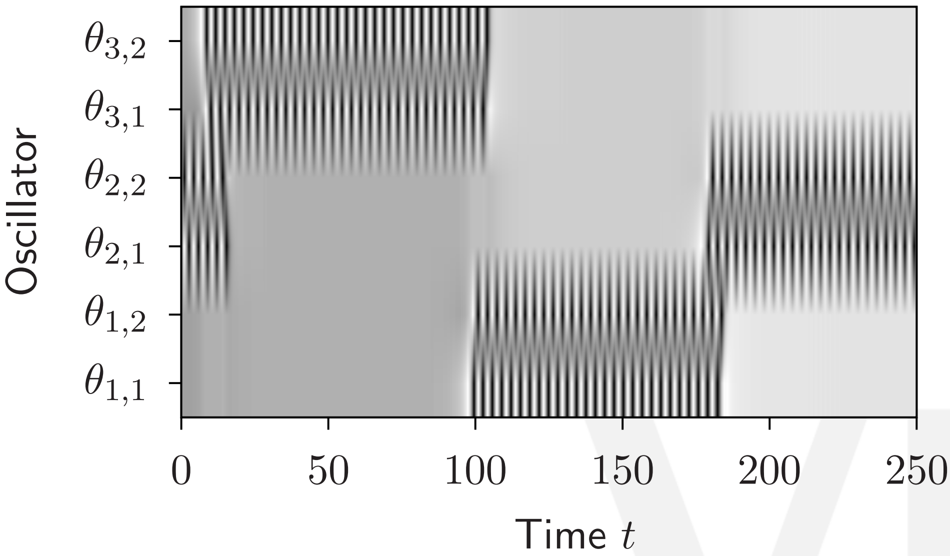
$\theta_{\sigma,k} \in [0, 2\pi)$

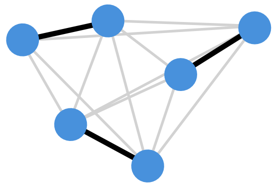
Q: Can one observe transitions of frequencies over time?

"Neuroscience"

 $\longrightarrow$ 







State of  :

$\theta_{\sigma,k} \in [0, 2\pi)$

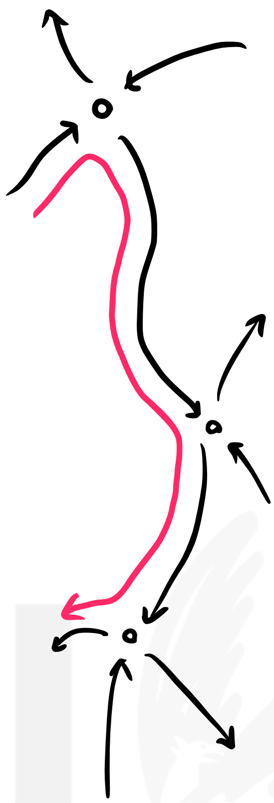
A: Yes, one may observe transitions of frequencies.

How to Get Transitions

Transitions induced by Heteroclinics Orbits

1. Have a finite collection of **saddles** A_q.
2. Suppose that the unstable manifold of A_q has a nontrivial intersection with the stable manifold of A_{q+1}—there are **heteroclinic connections**.
3. Impose additional **stability** conditions.

Dynamics: **Transitions from one saddle to the next along the cycle/network.**



Robust heteroclinic cycles/networks may arise in

- Lotka–Volterra type systems,
- Systems with symmetries.

Field (1996). Lectures on Bifurcations, Dynamics and Symmetry.

Coupled Populations with Higher-Order Interactions

Phase reduction, $\hat{g}_2$ higher harmonics, $\hat{g}_3, \hat{g}_4, \hat{g}_5$ one harmonic

$$
\begin{aligned}
\dot{\theta}_k = \omega &+ \sum_{j=1}^{Q} \hat{g}_2(\theta_j - \theta_k) + \sum_{j,l=1}^{Q} \hat{g}_3(\theta_j + \theta_l - 2\theta_k) \\
&+ \sum_{j,l=1}^{Q} \hat{g}_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{Q} \hat{g}_5(\theta_j + \theta_l - \theta_m - \theta_k)
\end{aligned}
$$

CB (2018). Phys Rev E, 97(5), 050201(R).

Coupled Populations with Higher-Order Interactions

Phase reduction, $\hat{g}_2$ higher harmonics, $\hat{g}_3, \hat{g}_4, \hat{g}_5$ one harmonic

$$\dot{\theta}_k = \omega + \sum_{j=1}^{Q} a_2^{(jk)} \hat{g}_2(\theta_j - \theta_k) + \sum_{j,l=1}^{Q} a_3^{(ljk)} \hat{g}_3(\theta_j + \theta_l - 2\theta_k)$$
$$+ \sum_{j,l=1}^{Q} a_4^{(ljk)} \hat{g}_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{Q} a_5^{(mljk)} \hat{g}_5(\theta_j + \theta_l - \theta_m - \theta_k)$$

CB (2018). Phys Rev E, 97(5), 050201(R).

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Special case: $M = 3$ populations of $N = 2$ oscillators, $j = 3 - k$

$$\begin{aligned}\dot{\theta}_{\sigma,k} = {} & \sin(\theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) + r\sin(2(\theta_{\sigma,j} - \theta_{\sigma,k} + \alpha)) \\ & - K\cos(\theta_{\sigma-1,1} - \theta_{\sigma-1,2} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\ & - K\cos(\theta_{\sigma-1,2} - \theta_{\sigma-1,1} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\ & + K\cos(\theta_{\sigma+1,1} - \theta_{\sigma+1,2} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\ & + K\cos(\theta_{\sigma+1,2} - \theta_{\sigma+1,1} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha)\end{aligned}$$

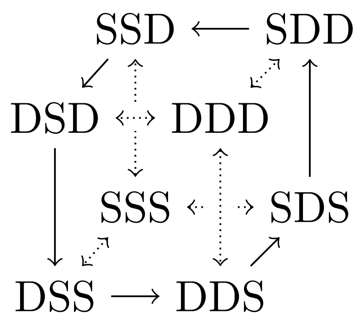
The system is T^M equivariant: one phase-shift symmetry per population.

CB (2018). Phys Rev E, 97(5), 050201(R).

Transitions of Localized Frequency Synchrony

Theorem

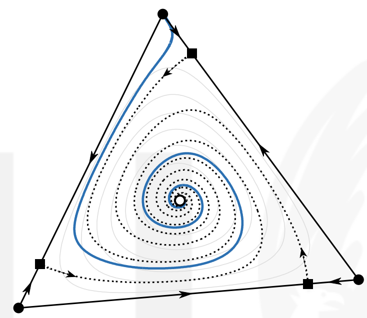
For $M = 3$ populations with $N = 2, 3$ oscillators, there are coupling parameters such that there is a robust and dissipative heteroclinic cycle between distinct patterns of localized frequency synchrony.



Idea of proof

1. No coupling: Separate frequencies of S, D.
2. Ensure stability of $\mathrm{DSS}, \mathrm{DDS}, \ldots$.
3. Heteroclinic connections, e.g., in $\mathrm{D}\psi\mathrm{S}$.

$\mathrm{D}\psi\mathrm{S}$ **for** $N = 3$:

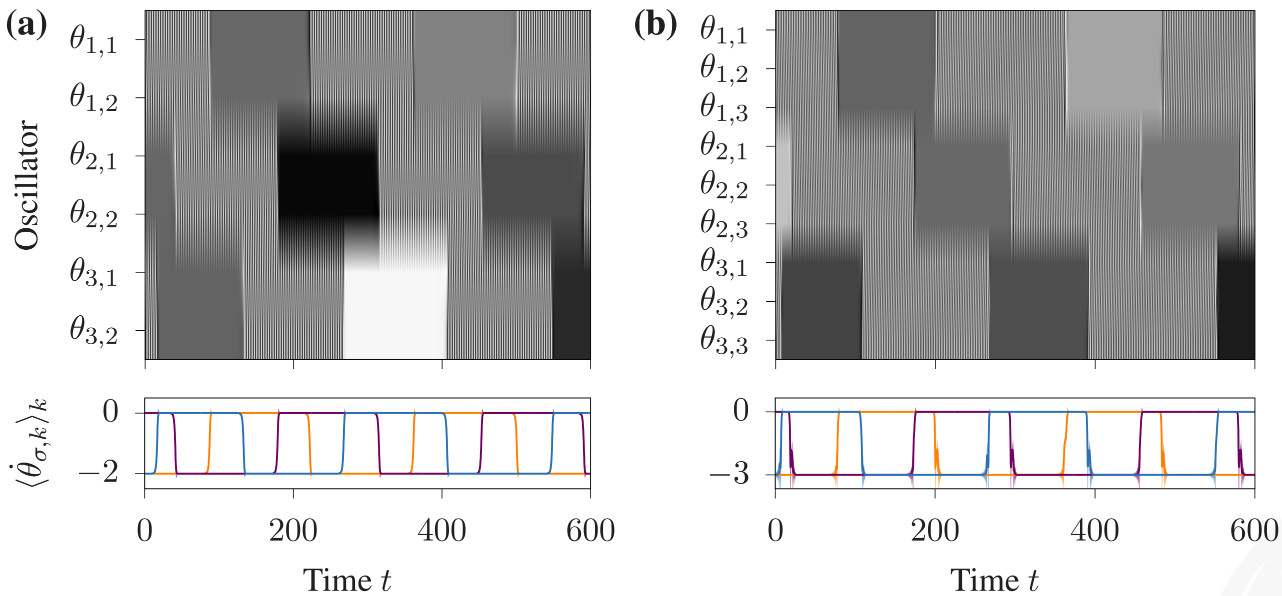


$\mathrm{D}\psi\mathrm{S}$ **for** $N = 2$:

CB (2018). Phys Rev E, 97(5), 050201(R). CB (2019) J Nonlin Sci, 29(6):2547–2570.

Heteroclinic Cycles in Action!

Dynamics of $M = 3$ populations of $N = 2, 3$ oscillators.



Project (J. Mujica, VU): Bifurcations of heteroclinic cycles under forced symmetry breaking.

Project (T. Böhle, TUM): Dynamics of the mean-field limit.

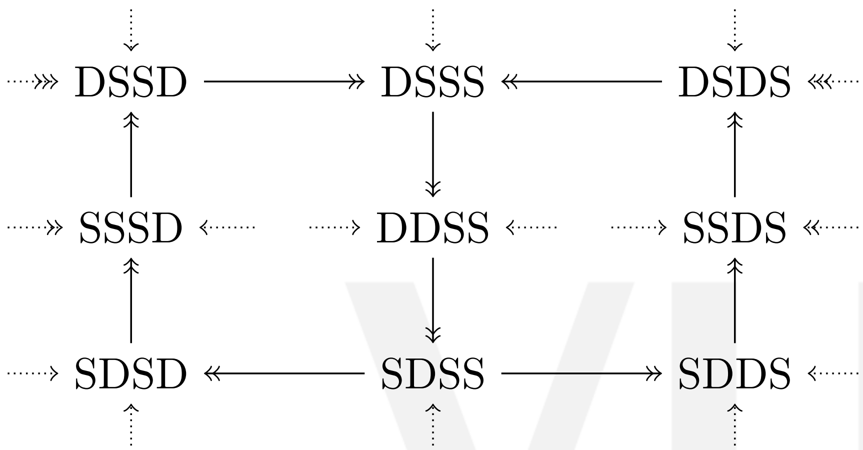
CB (2019) J Nonlin Sci, 29(6):2547–2570.

From Heteroclinic Cycles to Networks

Dynamics of $M = 4$ populations of $N = 2$ oscillators.

Theorem

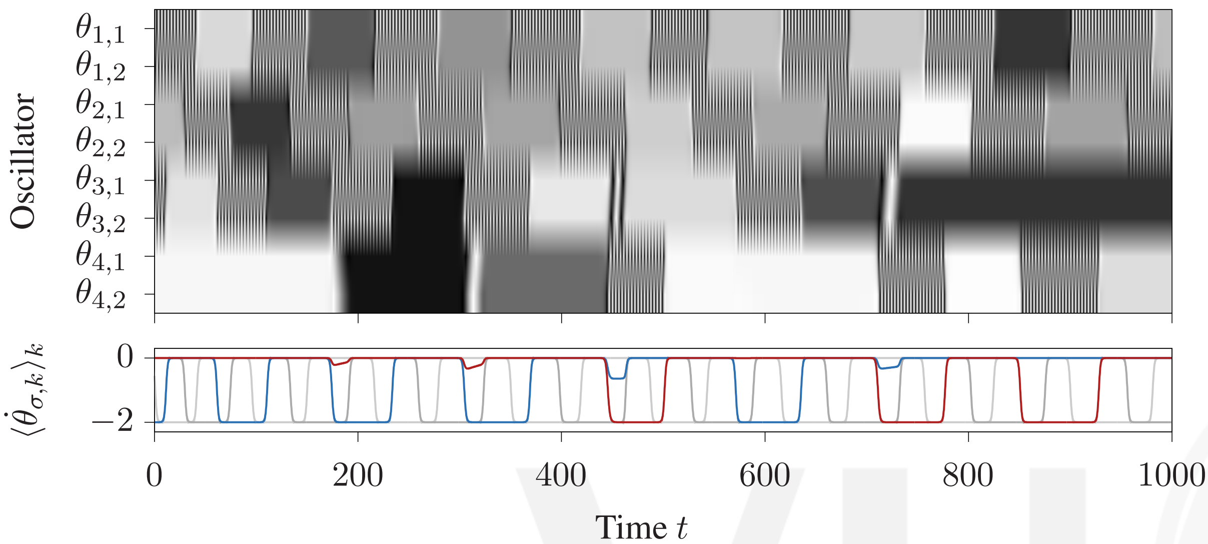
Coupled populations of phase oscillators support the heteroclinic network between distinct patterns of frequency synchrony below which contains two cycles.



CB and A Lohse (2019) J Nonlin Sci, 29(6):2571–2600.

More Action!

Dynamics of $M = 4$ populations of $N = 2$ oscillators.



CB and A Lohse (2019) J Nonlin Sci, 29(6):2571–2600.

Modular Networks

Phase Reduction

Weakly coupled nonlinear oscillators

$\downarrow$

Phase oscillator network

Synchronization Engineering

Nonlinear oscillator network with state x_k

$$\dot{x}_k = F(x_k) + \varepsilon \sum_{j=1}^{N} G_{kj}(x_j, x_k)$$

$\uparrow$

Calculate feedback parameters for $h(x)$ to match phase reduction.

$\uparrow$

Apply (delayed) **feedback** $p_k(x)$ to oscillators with known PRC $Z(\phi)$

$$\dot{x}_k = F(x_k) + Kp_k(x) \qquad p_k(t) = \sum_{j=1}^{N} K_{kj} h(x(t-\tau))$$

$\uparrow$

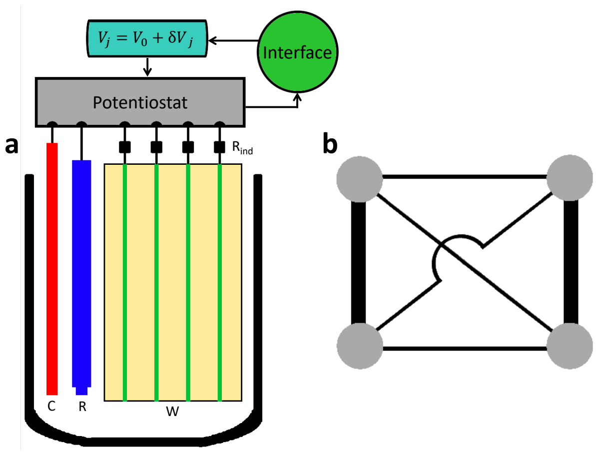
Phase oscillator network with state θ_k

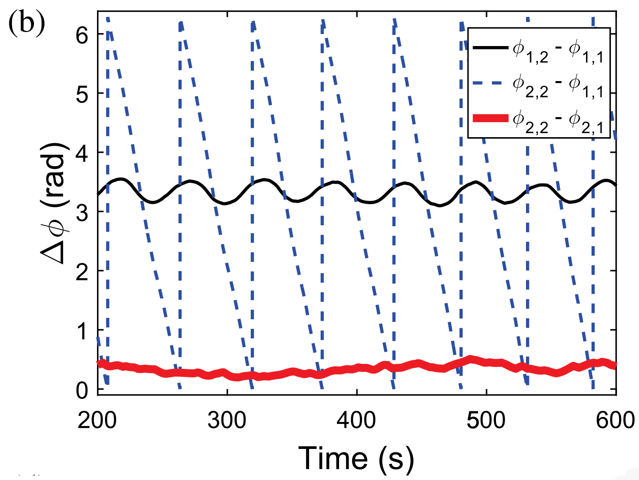
$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} g_{kj}(\theta_j - \theta_k)$$

H Kori, C G Rusin, I Z Kiss, and J L Hudson (2008). Chaos, 18(2), 26111

Experiments

Pairwise Interactions





Nonpairwise Interactions

Project (B. Liefting, Exeter). Generalization of Synchronization Engineering to Networks with Higher-Order Interactions.

CB, M Sebek, and I Z Kiss (2017). Phys Rev Lett, 119(16), 168301.

Conclusions and outlook

Conclusions

- **Higher-order interactions** yield interesting phase dynamics.
- **Identical oscillators** can give rise to **distinct frequencies** through network interactions.
- **Transitions of frequencies** can arise through heteroclinic cycles and networks.
- Describe such phenomena **mathematically** but can be seen in **experiments**

Outlook (i.e., more questions)

- Experimental realization of frequency transitions?
- Rigorous analysis of what happens as symmetries are broken.

Thank you for your attention!

References
Get in touch!

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Engineering and Physical Sciences Research Council

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