A non-commutative approach to the topology of circle and sphere bundles

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Algebra is but written geometry and geometry is but figured algebra.

Sophie Germain (1776–1831)

The duality between algebra and geometry dates back to the work of Descartes: coordinate system.

Algebraic geometry in the early XX century: Noether, Hilbert (Nullstellensatz, Basissatz):

$$\text{polynomial equations} \longleftrightarrow \text{algebraic variety}$$

Brought forward by Grothendieck.

For X a compact Hausdorff space, consider

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

The set $C(X)$ comes with

- vector space structure: for $f, g \in C(X)$ and $\lambda \in \mathbb{C}$

$$(\lambda f + g)(x) := \lambda f(x) + g(x), \qquad \forall x \in X;$$

- commutative product: for $f, g \in C(X)$:

$$(fg)(x) := f(x)g(x), \quad \forall x \in X;$$

- unit: the function identically equal to 1; and
- an involution $* : C(X) \to C(X)$ given by

$$f^*(x) := \overline{f(x)}.$$

There is a natural norm on the space $C(X)$, given by

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{1}$$

with respect to which $C(X)$ is *a Banach *-algebra*.

The norm satisfies

$$\boxed{\|f^* f\| = \|f\|^2.}$$

$C(X)$ is a *commutative* C^**-algebra*.

Example

Let X consist of n-points. $C(X) \simeq \mathbb{C}^n$ with the usual vector space structure, coordinate-wise multiplication and complex conjugation, and norm

$$\|(z_1, \dots z_n)\|^2 = \max\{\overline{z_i} z_i \; | i = 1, \dots, n\}$$

Any point $P \in X$ can be thought of as a functional

$$\sigma_P : C(X) \to \mathbb{C}, \quad \sigma_P(f) := f(P),$$

and it satisfies

$$\sigma_P(fg) = \sigma_P(f)\sigma_P(g), \quad \sigma_P(1) = 1,$$

i.e. σ_P is a *character* (also, a *pure state*).

All characters on $C(X)$ are of this form and the set of characters $\Sigma(C(X))$ is *homeomorphic* to X.

Theorem (Gelfand Duality)

*Let A be a commutative unital C^*-algebra. Then there is a *-isomorphism*

$$\boxed{A \simeq C(\Sigma(A))}$$

of commutative C^*-algebras.

Definition

A C*-algebra is a Banach *-algebra A with the property that

$$\boxed{\|a^* a\| = \|a\|^2},$$

for all $a \in A$.

Some examples

- The algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices with conjugate transpose and the operator norm

$$\|A\| = \sup_{x \in \mathbb{C}^n, \|x\|=1} \|Ax\|;$$

- The algebra $B(H)$ of bounded operators on a Hilbert space, with operator adjoint, and operator norm

$$\|A\| = \sup_{x \in H, \|x\|=1} \|Ax\|;$$

$B(H)$ is the prototypical example of C*-algebra.

Theorem (Gelfand–Naimark–Seagal)

Let A be a C-algebra. Then there exist a Hilbert space H and an injective $*$-homomorphism $\pi : A \to B(H)$.*

Every C*-algebra can be embedded into the bounded operators on a Hilbert space.

Idea

Motivated from Gelfand duality, look at noncommutative C*-algebras of operators as algebras of functions on some *noncommutative space*.

The circle:

$$S^1 := \{z \in \mathbb{C} \mid \bar{z}z = 1\}.$$

The C*-algebra $C(S^1)$ is the closure of the *Laurent polynomials*

$$\frac{\mathbb{C}[\zeta, \bar{\zeta}]}{\langle \bar{\zeta}\zeta = 1 \rangle}.$$

We represent $C(S^1)$ via multiplication operators on the Hilbert space

$$H = L^2(S^1) \simeq \ell^2(\mathbb{Z}).$$

Under this isomorphism, multiplication by $e^{2\pi \mathrm{i}\theta}$ is mapped to the bilateral shift

$$U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}.$$

$C(S^1)$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{Z}))$ that contains the *unitary* U.

Now instead consider the Hilbert space $\ell^2(\mathbb{N})$ and the shift operator

$$T(e_n) = (e_{n+1})$$

Its adjoint is not invertible

$$T^*(e_n) = \begin{cases} e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

The *Toeplitz algebra* $\mathcal{T}$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains T. It is not commutative since

$T^*T = \mathrm{Id}$ and $TT^* = 1 - P_{\ker(T^*)}$.

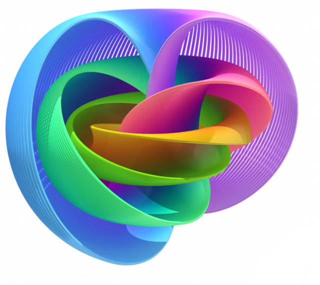
Elements of $\mathcal{T}$ commute up to compact operators:

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

The spectrum $\Sigma(\mathcal{T})$ (defined as the set of pure states) is the disk $\mathbb{D} \subseteq \mathbb{C}$.

The algebra $C(S^1)$ is the "boundary" of a noncommutative disk.

Topology	**Operator algebra**
topological space	C*-algebra
point	pure state
vector bundle	finitely generated projective module
topological K-theory	operator K-theory
Hermitian vector bundle	finitely generated projective *Hilbert* module
circle bundle	Cuntz–Pimsner algebra
sphere bundle	Subproduct Systems



Principal circle bundle

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

Look at S^3 inside $\mathbb{C}^2$:

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \overline{z_1} z_1 + \overline{z_2} z_2 = 1\}.$$

Circle action defined component-wise: for every $\lambda \in S^1$,

$$\alpha_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2).$$

The orbit space is the two sphere S^2.

In physics: connections on the Hopf bundle describe magnetic monopole potentials.

The Hopf projection $\pi : S^3 \to S^2$ dualises to an inclusion of C*-algebras

$$C(S^2) \overset{\iota}{\hookrightarrow} C(S^3).$$

Circle action on $C(S^3)$, such that $C(S^2)$ is the fixed point algebra. The coordinate algebra

$$C(S^3) \supseteq \mathcal{O}(S^3) := \frac{\mathbb{C}[z_1, z_2, \overline{z_1}, \overline{z_2}]}{\langle \overline{z_1} z_1 + \overline{z_2} z_2 = 1 \rangle}$$

admits a vector space decomposition

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

where each $\mathcal{L}_n$ is the space of elements of $\mathcal{O}(S^3)$ that transform under the circle action as

$$\phi \mapsto \lambda^{-n} \phi, \qquad \forall \lambda \in S^1$$

Each $\mathcal{L}_n$ is a bimodule over $\mathcal{L}_0 \simeq \mathcal{O}(S^2)$ and it is finitely generated projective.

The condition that the bundle is *principal* translates into the *algebraic* condition that the grading

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

is *strong*, i.e.

$$\mathcal{L}_n \otimes_{\mathcal{L}_0} \mathcal{L}_m \simeq \mathcal{L}_{n+m}.$$

This is in turn equivalent to invertibility of the module $\mathcal{L}_1$:

$$\mathcal{L}_1 \otimes_{\mathcal{L}_0} \mathcal{L}_{-1} \simeq \mathcal{L}_0 \simeq \mathcal{L}_{-1} \otimes_{\mathcal{L}_0} \mathcal{L}_1.$$

The Peter–Weyl decomposition allows to decompose the coordinate algebra of a circle bundle into *sums of powers of line bundles* and to characterise principal circle bundles. Many C*-algebras have this structure.

Hilbert modules generalize the notion of Hilbert space with the field $\mathbb{C}$ replaced by a C*-algebra B.

A Hilbert module is a pair $(E, \langle\ ,\ \rangle_B)$, where

- E is a right B-module with an Hermitian B-valued inner product; and
- E is complete in the norm
$$\|\xi\|^2 := \|\langle \xi, \xi \rangle_B\|^2.$$

Operations on Hilbert modules: direct sums, tensor products.

The adjointable operators

$$\mathrm{End}^*(E) := \{T : E \to E \mid \exists T^* : E \to E : \langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle\},$$

form a C*-algebra.

Define the C*-algebraic dual

$$E^* := \{\lambda_\xi, \xi \in E \mid \lambda_\xi(\eta) = \langle \xi, \eta \rangle\} \subseteq \mathrm{Hom}^*(E, B).$$

Let E be a finitely generated projective Hilbert bimodule over a unital C*-algebra B.
We say that E is a *self-Morita equivalence* over B if

$$E \otimes_B E^* \simeq B \simeq E^* \otimes_B E.$$

Example

Let $B = C(X)$. Then $E = \Gamma(L)$, the module of sections of a Hermitian line bundle $L \to X$ is a self-Morita equivalence over B.

Out of internal tensor products, construct

$$\mathcal{F}(E) := B \oplus \bigoplus_{n \geq 1} E^{\otimes n}$$

For every $\xi \in E$ define the *shift operators* by

$$T_\eta(\xi_1 \otimes \cdots \xi_n) = \eta \otimes \xi_1 \otimes \cdots \xi_n, \quad T_\eta b = \eta \cdot b.$$

They are adjointable operators on $\mathcal{F}(E)$.

Definition

The *Toeplitz algebra* $\mathcal{T}_E$ is the smallest C*-subalgebra of $\mathrm{End}^*(\mathcal{F}(E))$ that contains all the shifts.

If E is a self-Morita equivalence bimodule, we can define the two-sided Fock module

$$\mathcal{F}_{\mathbb{Z}}(E) := \bigoplus_{n \in \mathbb{Z}} E^{(n)}$$

where $E^{(n)} := E^{\otimes n}$ for $n > 0$, $E^{(0)} = B$ and $E^{(n)} := (E^*)^{\otimes n}$ for $n < 0$.
On $\mathcal{F}_{\mathbb{Z}}(E)$ we consider bilateral shift operators S_ξ, $\xi \in E$.

Definition

The *Cuntz–Pimsner algebra* of E, denoted $\mathcal{O}_E$, is the smallest C^*-subalgebra of $\mathrm{End}^*(\mathcal{F}_{\mathbb{Z}}(E))$ which contains all the bilateral shift operators.

We have an exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(E)) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0.$$

Both $\mathcal{T}_E$ and $\mathcal{O}_E$ come endowed with a circle action.
We denote by $\mathcal{O}_E^\gamma$ the fixed point algebra for this action.

Proposition (A.–Rennie)

E is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^\gamma \simeq B$.

Theorem (A.–Kaad–Landi)

Pimsner algebras of self-Morita equivalences are quantum principal circle bundles.

Examples: q-deformations

The six-term exact sequence in **operator** K-theory

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-[X]} & K_0(A) & \xrightarrow{j_*} & K_0(\mathcal{O}_X) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_X) & \xleftarrow[j_*]{} & K_1(A) & \xleftarrow[1-[X]]{} & K_1(A) \end{array},$$

is a noncommutative analogue of the **topological** K-theory Gysin sequence for a circle bundle $P \to X$ coming from the Hermitian line bunlde L.

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{1-[L]} & K^0(X) & \xrightarrow{j^*} & K^0(P) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K^1(P) & \xleftarrow[j^*]{} & K^1(X) & \xleftarrow[1-[L]]{} & K^1(X) \end{array},$$

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

Cuntz Pimsner algebras of (injective) C*-correspondences.

$$0 \longrightarrow \mathcal{K}(F(E)) \xrightarrow{j} \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0.$$

Arveson's Toeplitz extensions for odd-dimensional spheres.

$$0 \longrightarrow \mathcal{K}(H^2_d) \xrightarrow{j} \mathcal{T}_d \xrightarrow{\pi} C(S^{2d-1}) \longrightarrow 0.$$

All these are examples of the *defining extensions* for Cuntz–Pimsner algebras of **subproduct systems** (Shalit and Solel 2009, Viselter 2012).

Let $d \in \mathbb{N}_0$, and $z_0, \ldots, z_{d-1}$ commuting variables, and consider the space of polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$.
For $z = (z_0, ..., z_{d-1})$ and every multi-index $\alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{N}_0^d$ we write

$$z^\alpha = z_0^{\alpha_0} \cdots z_{d-1}^{\alpha_{d-1}}.$$

The Drury–Arveson space H^2_d is a completion of the polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$, w.r.t. the inner product

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha!}{|\alpha!|}$$

It can be identified with the space of holomorphic functions $f : \mathbb{B}^d \subseteq \mathbb{C}^d \to \mathbb{C}$ which have a power series $f(z) = \sum_\alpha c_\alpha z^\alpha$ satisfying

$$\|f\|_d^2 := \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha!|} < \infty.$$

Clearly, $H^2_d \simeq \mathbb{F}_{\mathrm{sym}}(\mathbb{C}^d) := \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathbb{C}^d)$, the d-symmetric Fock space.

On H^2_d, we consider the d-shift, a d-tuple of multiplication operators given by

$$Mz = (M_{z_{\mathbf{0}}}, \dots, M_{z_{d-\mathbf{1}}}).$$

Through $H^2_d \simeq \mathbb{F}_{\mathrm{sym}}(\mathbb{C}^d)$, the shift operator is identified with a compression of the shift on the full Fock space, that we denote by $T = (T_0, \dots, T_{d-1})$.
The d-shift satisfies the following properties:

- T is commuting: $T_i T_j = T_j T_i$.
- $\sum_{i=0}^{d-1} T_i T_i^* = 1 - P_{\mathbb{C}}$
- T is essentially normal:

$$T_i^* T_j - T_j T_i^* = (1 + N)^{-1}(\delta_{ij} 1 - T_j T_i^*),$$

where N is the number operator: $N\xi = n\xi$ for $\xi \in \mathrm{Sym}^n(\mathbb{C}^d)$.

Theorem (Arveson 1998)

Let $\mathbb{T}_d = C^*(1, T)$ be the C^*-algebra generated by the d-shift. We have an exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(H^2_d) \longrightarrow \mathbb{T}_d \longrightarrow C(S^{2d-1}) \longrightarrow 0 \;, \tag{2}$$

where $C(S^{2d-1})$ is the commutative C^*-algebra of continuous functions on the $(2d-1)$-sphere $S^{2d-1} = \partial\mathbb{B}^d \subseteq \mathbb{C}^d$.

Odd-dimensional sphere as "boundaries" of a noncommutative C*-algebras of operators.

Example of a *subproduct system* extension.

Work in progress with J. Kaad (SDU Odense)

For a given $n \geq 0$, consider the *irreducible* representation $\rho_n : SU(2) \to U(L_n)$. Where $L_n = (\mathbb{C}^2)^{\otimes_S n}$.
We define the determinant of the representation:

$$\det(\tau, H) = \{\xi \in H \otimes H \mid (\tau(g) \otimes \tau(g))\xi = \xi \quad \forall g \in SU(2)\}.$$

We inductively construct a family of Hilbert spaces where

- $E_0 = \mathbb{C}$;
- $E_1 = L_n$;
- $E_m := K_m^\perp \subseteq (L_n)^{\otimes m}$, where

$$K_m = \sum_{i=0}^{m-2} L_n^{\otimes i} \otimes D \otimes L_n^{\otimes(m-i-2)}, \qquad D := \det(\rho_n, L_n).$$

We construct the Fock space $F_E := \bigoplus_{m\geq 0} E_m(\rho_n, L_n)$.
We let $\{e_j\}_{j=0}^n$ denote the orthonormal basis for L_n and consider the associated Toeplitz operators:

$$T_i := T_{e_i} : F_E \to F_E \quad T_i(\zeta) := \iota_{1,m}^*(e_i \otimes \zeta), \quad \zeta \in E_m(\rho_n, L_n).$$

where $\iota_{1,m} : E_{m+1} \to E_1 \otimes E_m$, for $m \in \mathbb{N}_0$.

Definition

The Toeplitz algebra of the subproduct system $\mathbb{T}_E$ the unital C*-algebra generated by the Toeplitz operators.

It comes with a natural $SU(2)$-action so that we have an equivariant $SU(2)$-extension of C*-algebras:

$$0 \longrightarrow \mathbb{K}(F_E) \longrightarrow \mathbb{T}_E \xrightarrow{q} \mathbb{O}_E \longrightarrow 0. \tag{3}$$

Theorem (A–Kaad 2020)

Let $\mathbb{T}_E$ be the Toeplitz algebra of the $SU(2)$-product system of an irreducible representation. Then $\mathbb{T}_E$ and $\mathbb{C}$ are KK-equivalent (i.e. the same in K-theory and K-homology) in an $SU(2)$-equivariant way.

We have *Gysin*-type exact sequence

$$0 \longrightarrow K_1(\mathbb{O}) \xrightarrow{([F]\widehat{\otimes}_{\mathbb{K}(F)}\cdot)\circ\partial} K_0(\mathbb{C}) \xrightarrow{\mathbf{1}_{\mathbb{C}}-[L_n]+[\det(\rho_n,L_n)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(\mathbb{O}) \longrightarrow 0$$

for every $n \in \mathbb{N}$.

Note that the Euler class comprises of three terms, as we would expect classically!

- C^*-algebras provide an elegant setting for problems in geometry and topology.
- Within the NCG dictionary, Cuntz–Pimsner algebras are a model for circle bundles.
- Cuntz–Pimsner algebas of subproduct systems are suitable to encode spherical symmetries.