Synchrony and Phase Relations in Network Dynamics

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Parts of this are joint work with:

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Alessio Franci

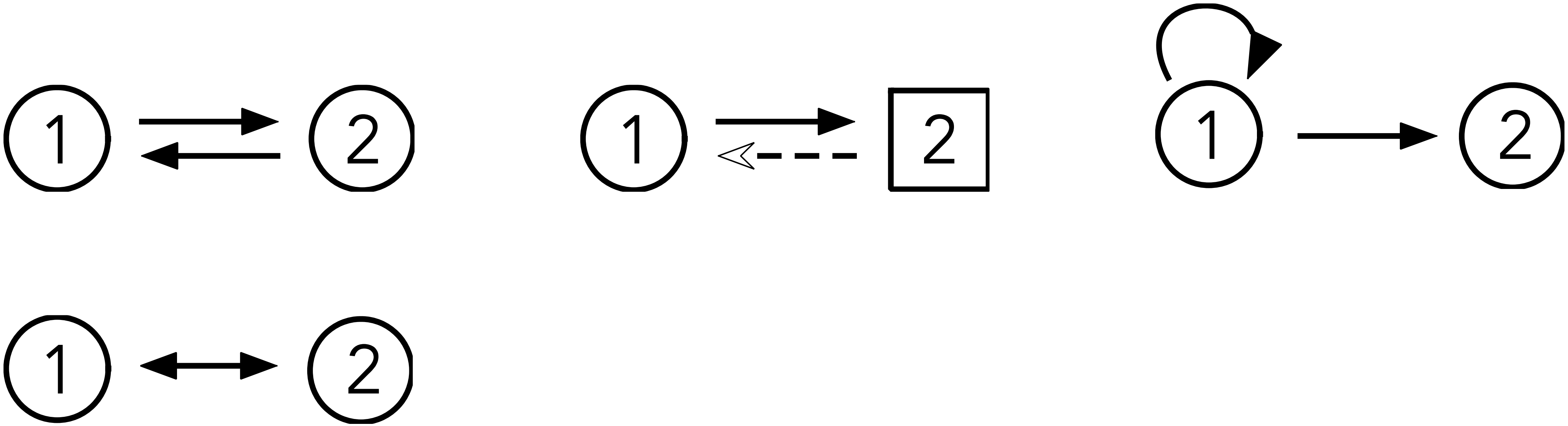
Martin Golubitsky

Anastasia Bizyaeva

Naomi Ehrich Leonard

Network

A directed graph whose nodes and arrows are assigned 'types'.



Network

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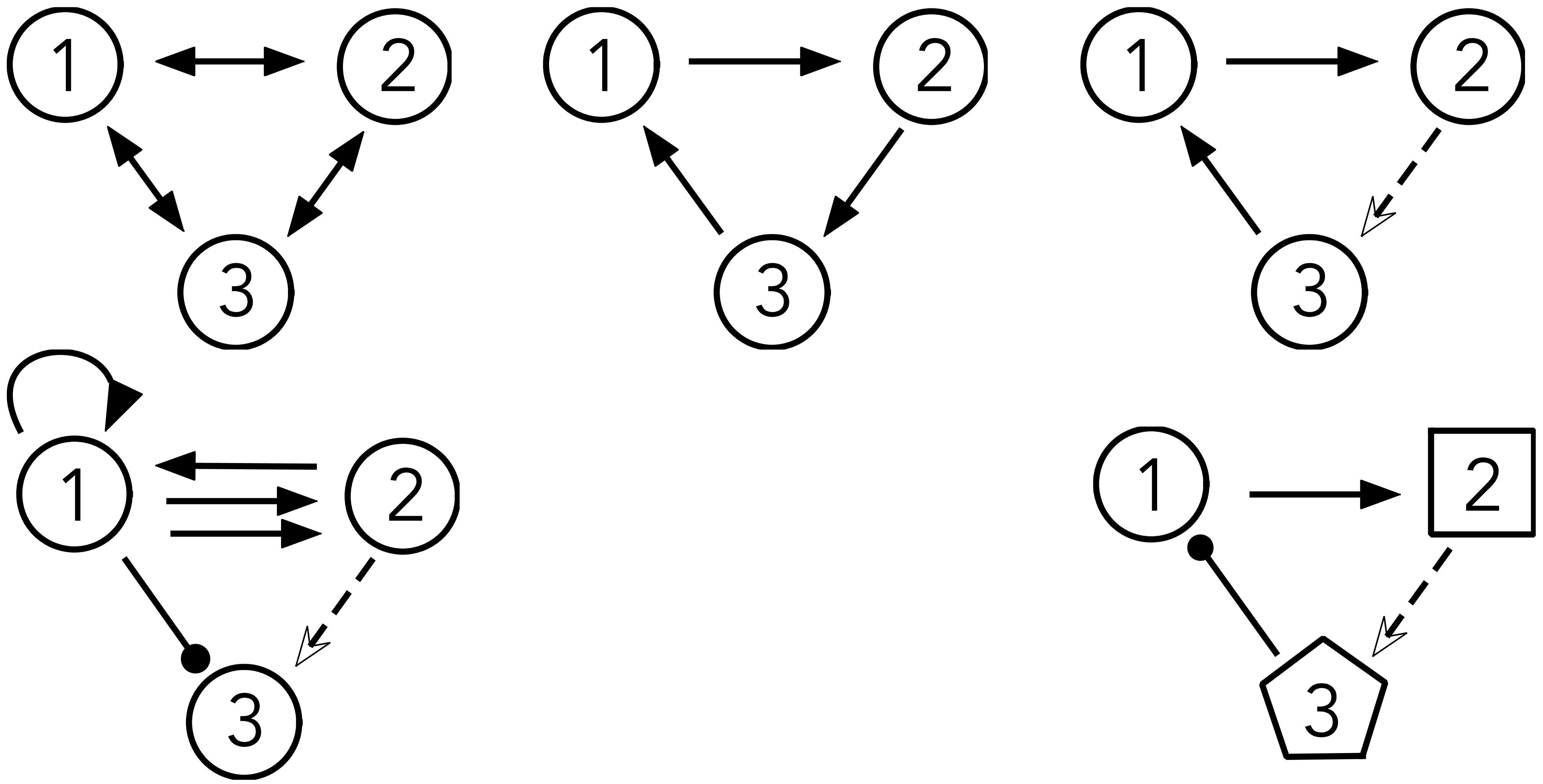


Node Space

Assign to each node a state space / phase space / node space R^k.

There are rules that require certain nodes to have the same node space. For technical reasons, this is not just 'same node type'. Same state type.

Other nodes might have the same node space for 'accidental' reasons. For example, all node spaces might be R.

The current convention is not quite the same as that in earlier literature.
(For good reasons which I won't go into.)

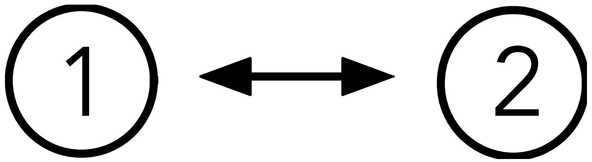
Admissible Maps / Vector Fields /ODEs

Dynamics that respects the network diagram and the classification into types.

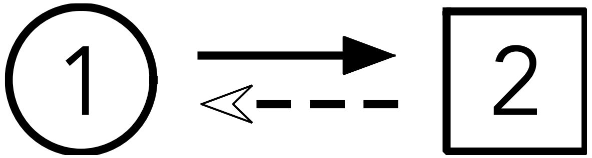
Node -> dynamical system
Node type -> internal dynamic (roughly)
Identical node symbols -> identical internal dynamics
(same formula, same parameters, only variables differ)

Arrow -> coupling between systems (input/output)
Arrow type -> type of coupling
Identical arrow types -> same kind of coupling
(same formula, same parameters, only variables differ)

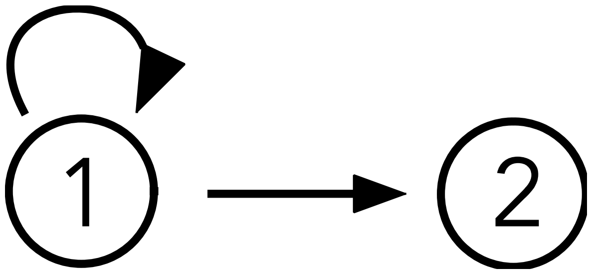
Examples of Admissible ODEs



$$\dot{x}_1 = f(x_1, x_2)$$
$$\dot{x}_2 = f(x_2, x_1)$$



$$\dot{x}_1 = f(x_1, x_2)$$
$$\dot{x}_2 = g(x_2, x_1)$$



$$\dot{x}_1 = f(x_1, x_1)$$
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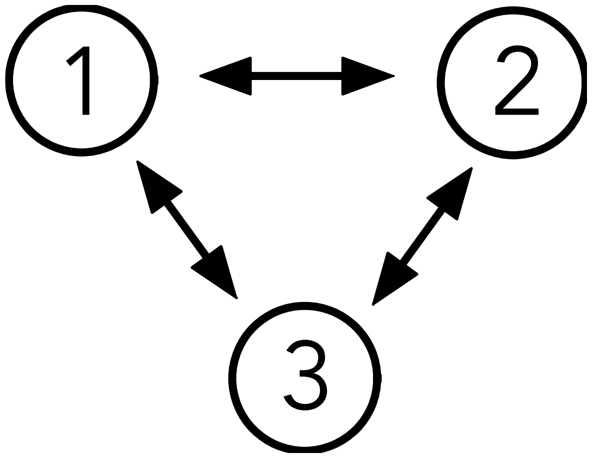
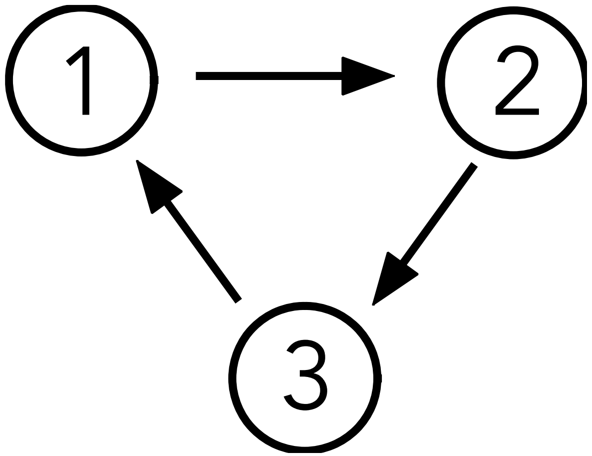
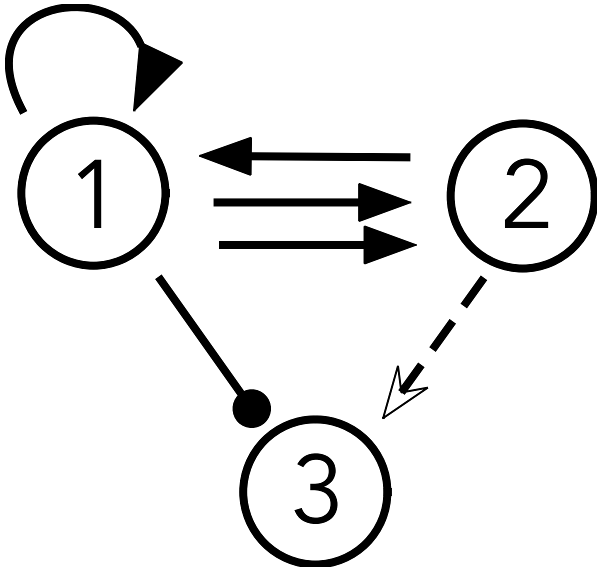
Examples of Admissible ODEs







$$\dot{x}_1 = f(x_1,\overline{x_2,x_3})$$
$$\dot{x}_2 = f(x_2,\overline{x_1,x_3})$$
$$\dot{x}_3 = f(x_3,x_1,x_2)$$

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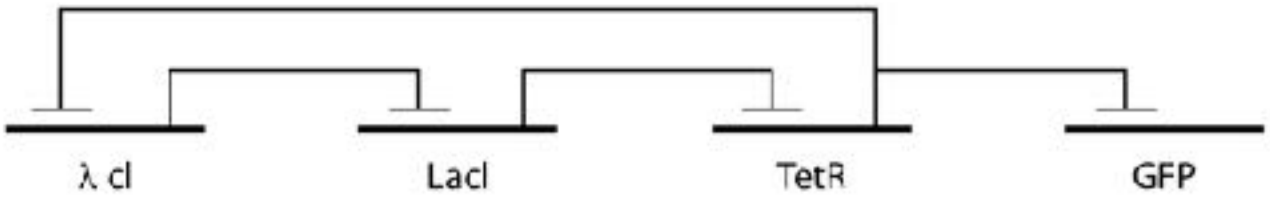
$$\dot{x}_1 = f(x_1,\overline{x_1,x_2})$$
$$\dot{x}_2 = f(x_2,\overline{x_1,x_1})$$
$$\dot{x}_3 = g(x_3,x_1,x_2)$$

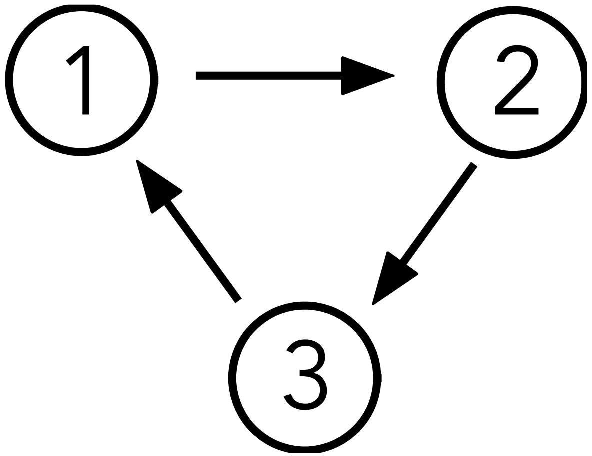
convention: first variable is node variable

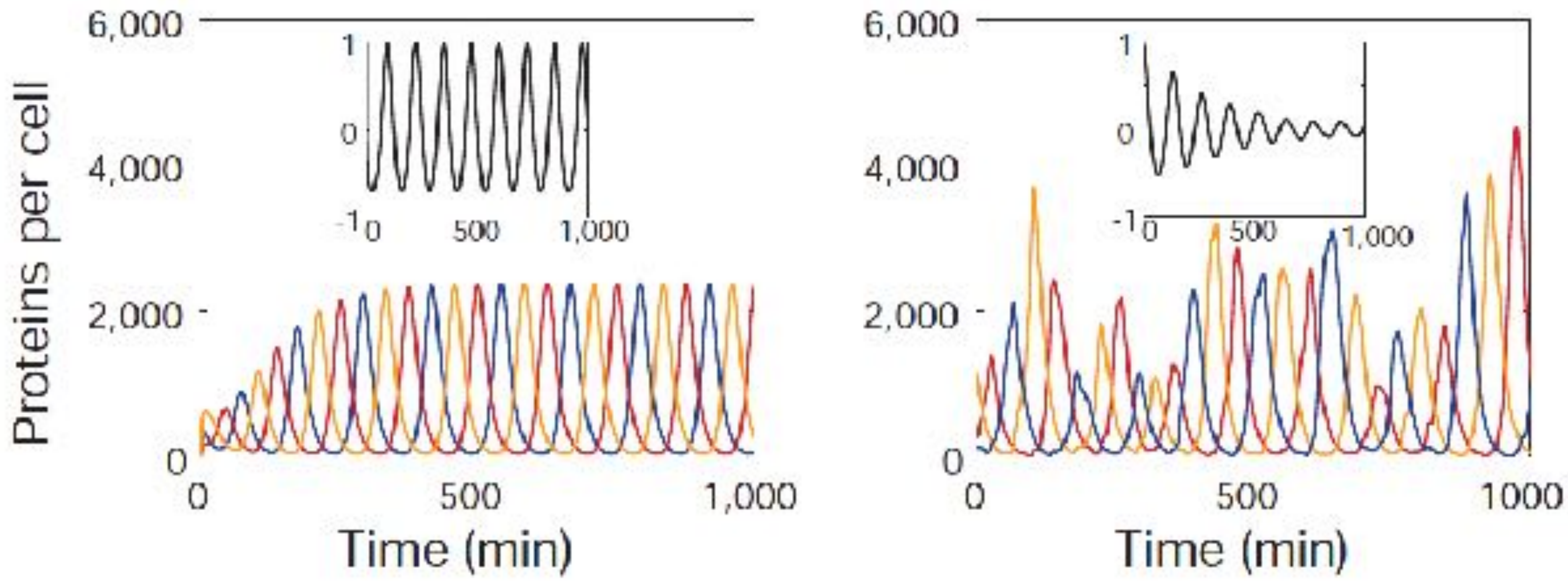
$\overline{\quad\quad}$ means 'symmetric in these variables'

Example: Repressilator

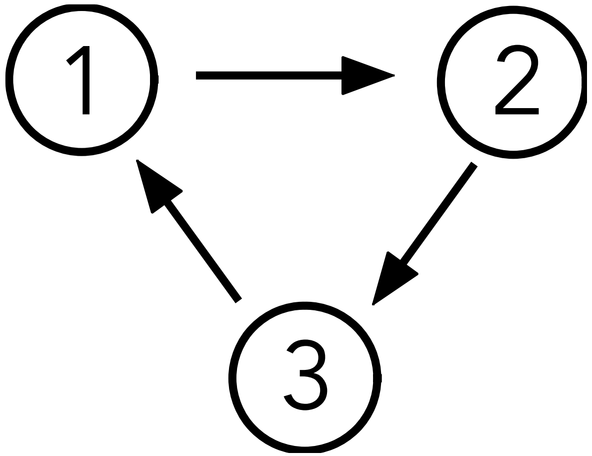
Synthetic Gene Regulatory Network





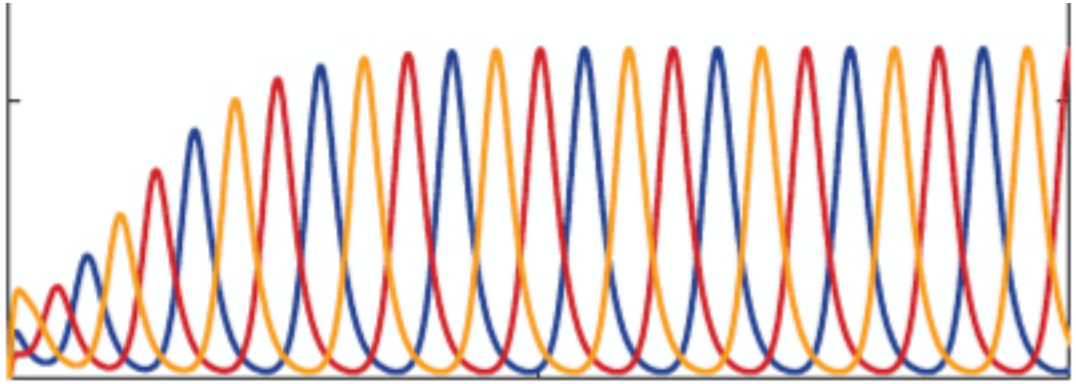


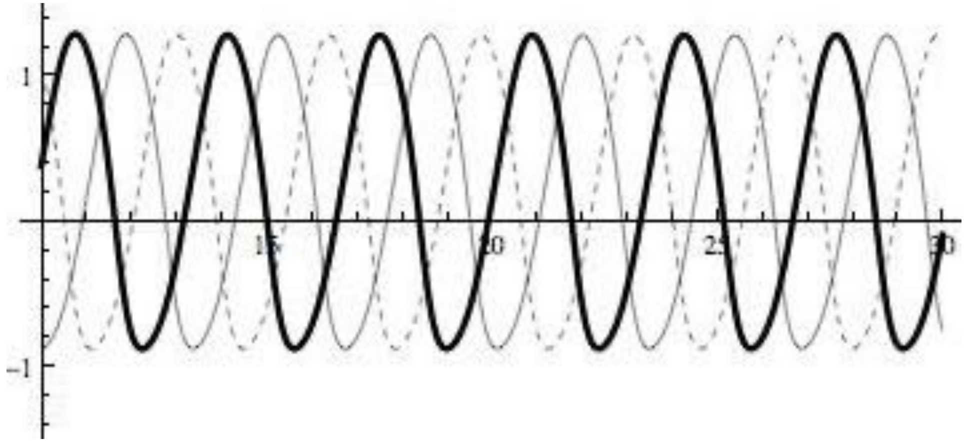
Example: Repressilator



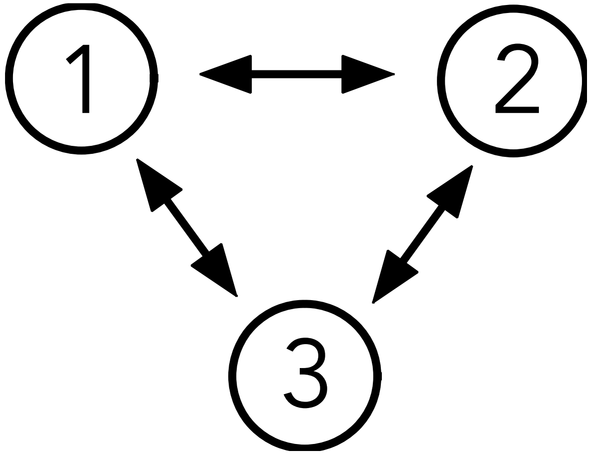
Rotating wave of 1/3-period phase shifts

Phase pattern





Synchrony and Phase Relations

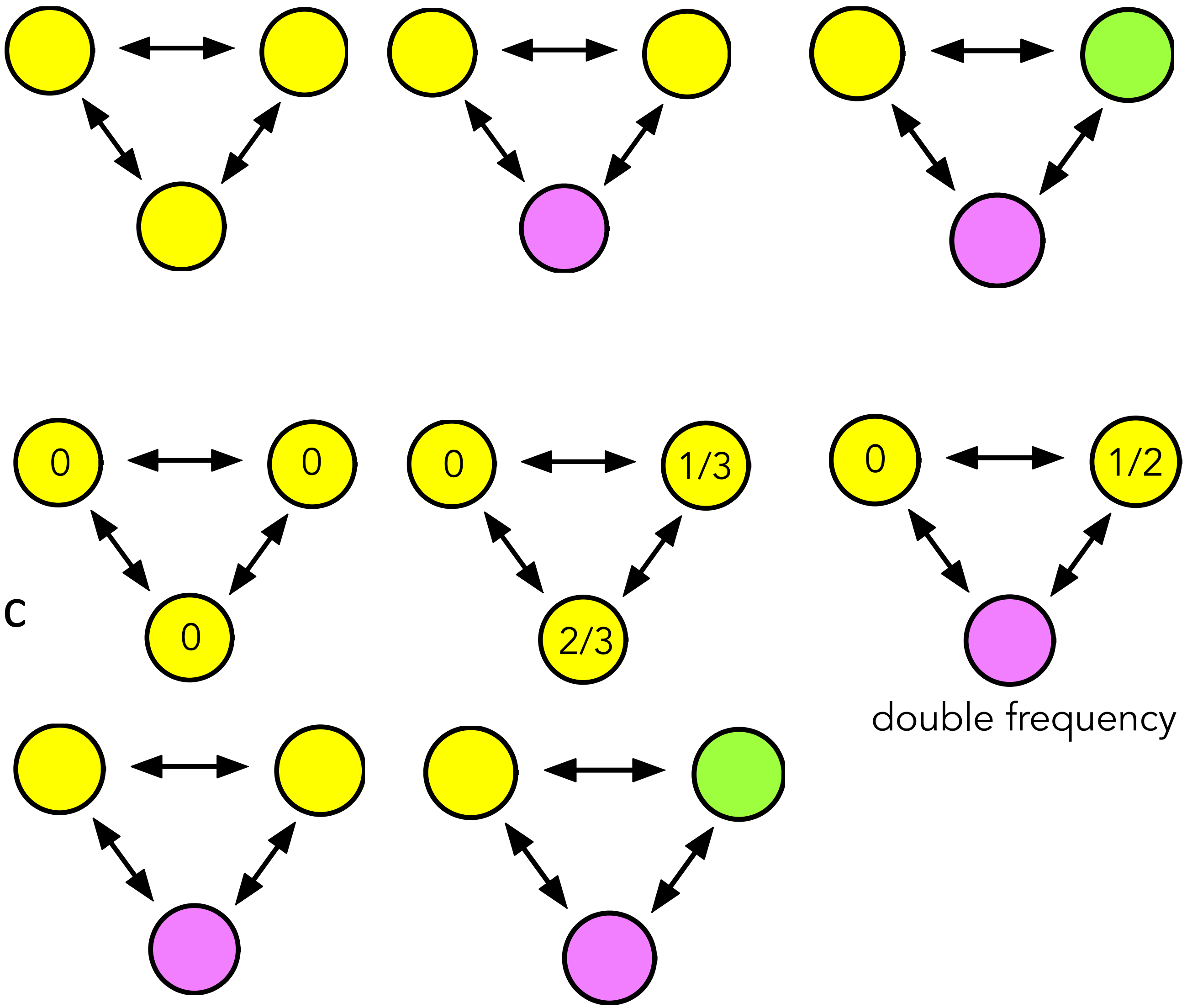


$$x^{\cdot}_1 = f(x_1, x_2, x_3)$$
$$x^{\cdot}_2 = f(x_2, x_1, x_3)$$
$$x^{\cdot}_3 = f(x_3, x_1, x_2)$$

symmetry group $\mathbf{D}_3 = \mathbf{S}_3$

steady

periodic

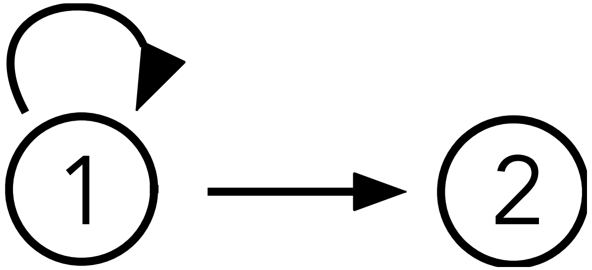


Synchrony and Phase Relations: Symmetry

Symmetry is sufficient to induce certain synchrony and phase patterns

Is it also necessary? NO!

Consider the ***synchrony pattern*** {{1,2}}



$$x_1(t) = x_2(t) \; [= x(t)]$$

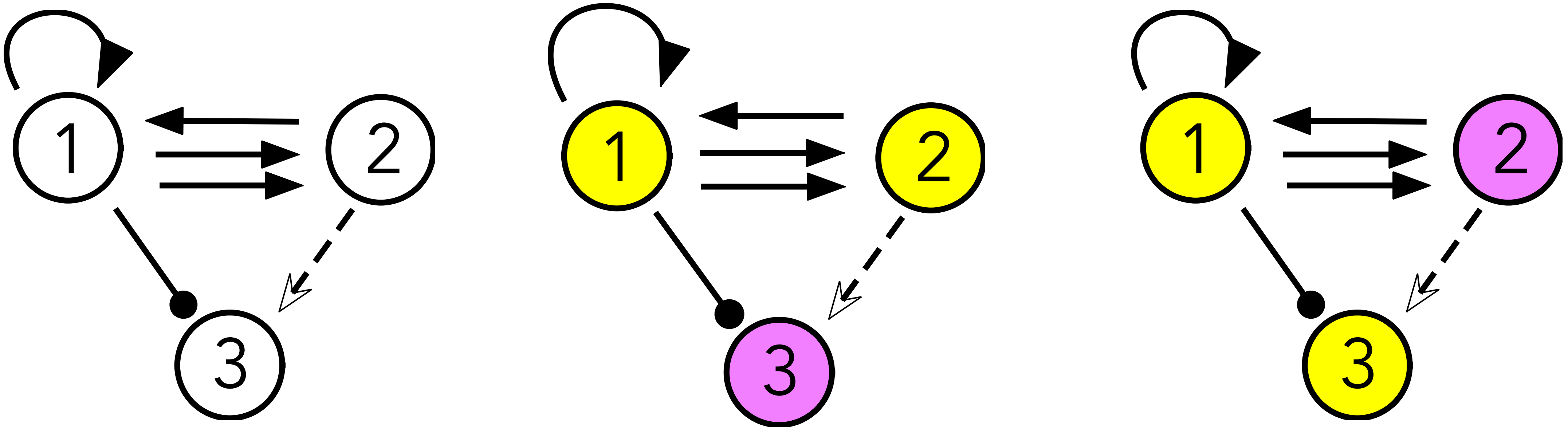
$$x^{\cdot} = f(x_1, x_1)$$
$$x^{\cdot} = f(x_2, x_1)$$

trivial symmetry

$$x^{\cdot} = f(x, x)$$
$$x^{\cdot} = f(x, x)$$

If node space has dimension ≥ 2, this can have a periodic solution for suitable f

Synchrony Patterns and Cluster Dynamics

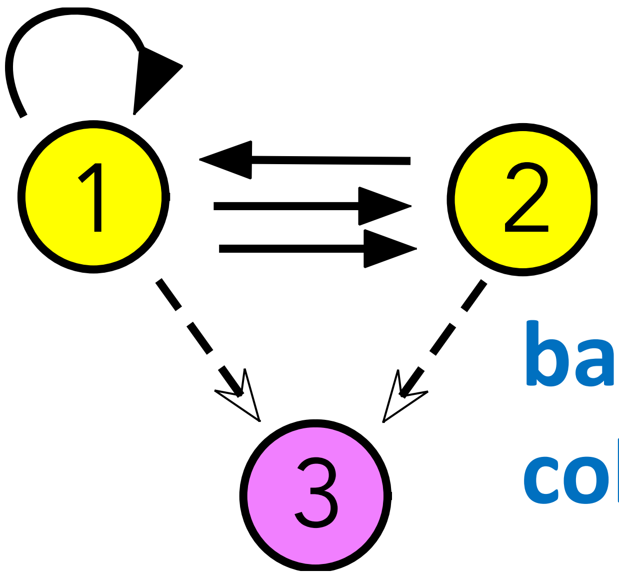


$$\dot{x}_1 = f(x_1, x_1, x_1, x_2)$$
$$\dot{x}_2 = f(x_2, x_1, x_1, x_1)$$
$$\dot{x}_3 = g(x_3, x_1, x_2)$$

$$\dot{x}_1 = f(x_1, x_1, x_1, x_1)$$
$$\color{red}{\dot{x}_1} = f(\color{red}{x_1}, x_1, x_1, x_1)$$
$$\dot{x}_3 = g(x_3, x_1, \color{red}{x_1})$$

$$\dot{x}_1 = f(x_1, x_1, x_1, \color{red}{x_1})$$
$$\dot{x}_2 = f(x_2, x_1, x_1, x_1)$$
$$\color{red}{\dot{x}_1} = g(\color{red}{x_1}, x_1, x_2)$$

Synchrony Patterns and Cluster Dynamics

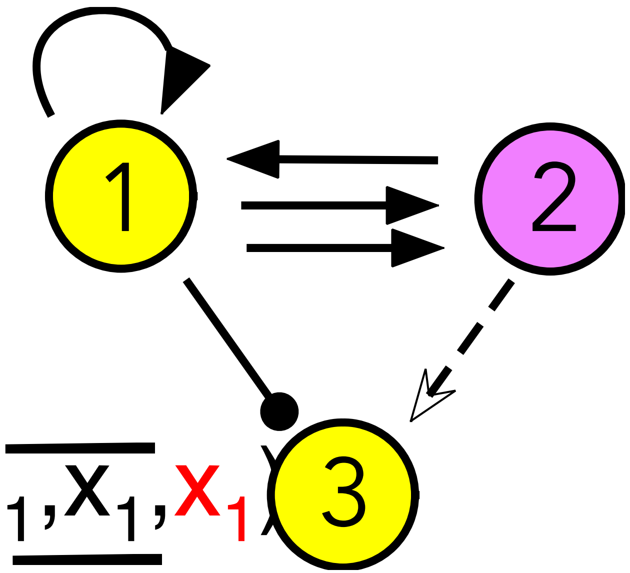


balanced colouring

$$\dot{x}_1 = f(x_1,\overline{x_1,x_1})$$

$$\dot{x}_3 = g(x_3,x_1,{\color{red}x_1})$$

any solution of this ODE gives
a solution of the original ODE with
synchrony pattern {{1,2},{3}}



$$\dot{x}_1 = f(\underline{\overline{x_1,x_1}},{\color{red}x_1})$$

$$\dot{x}_2 = f(x_2,x_1,x_1)$$

$${\color{red}\dot{x}_1} = g({\color{red}x_1},x_1,x_2)$$

unbalanced colouring

the first and third equations
conflict — **no solution**

Colourings and Balance

Let the set of nodes be C = {1,2 ..., n}.
A colouring is:

A set of colours K and a map k: C -> K that assigns a colour k(c) to each node c.

A partition of the set C. Parts = nodes with same colour.

An equivalence relation on C. Nodes related iff same colour.

All these are equivalent.

A colouring is balanced if nodes of the same colour have colour-isomorphic input sets.

Input arrows correspond by a bijection, which preserves arrow-types and colours of tail nodes.

Synchrony Spaces

Synchrony Space / Synchrony Subspace

Let $\bowtie$ be a colouring.

The synchrony space $\triangle_{\bowtie}$ is the space of all x = (x_c) where c is in C, such that

c and d have the same colour implies $x_c = x_d$

These are the states of the network system that have the synchrony pattern defined by the colouring.

Basic Theorems on Balanced Colourings

A synchrony space $\Delta_{\bowtie}$ is invariant under every linear admissible map if and only if $\bowtie$ is balanced.

The synchrony space $\Delta_{\bowtie}$ is flow-invariant (invariant under every admissible map) if and only if $\bowtie$ is balanced.

A vector subspace is flow-invariant if and only if it is $\Delta_{\bowtie}$ for some balanced coloring $\bowtie$.

A vector subspace may be invariant under every linear admissible map without being a synchrony space.

Quotient Networks

Given an admissible ODE

$$\dot{x} = f(x)$$

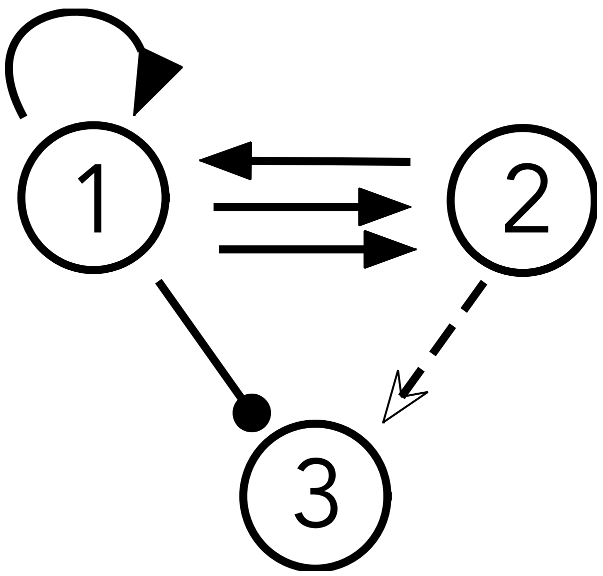
We can restrict f to any balanced synchrony space $\Delta_{\bowtie}$ to obtain a reduced ODE

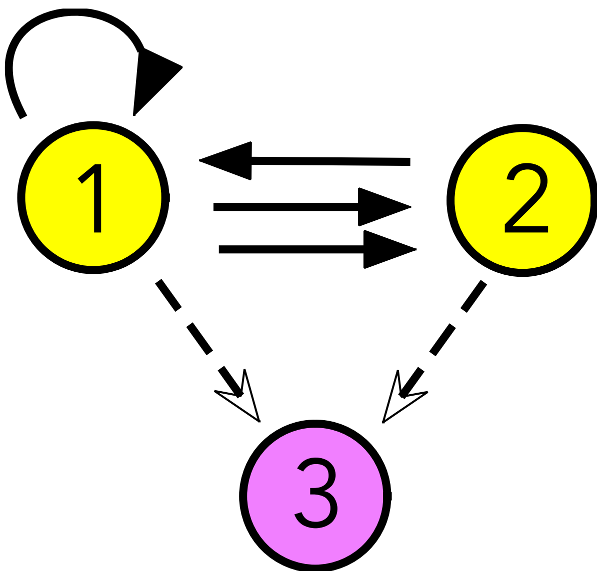
$$\dot{y} = g(y)$$

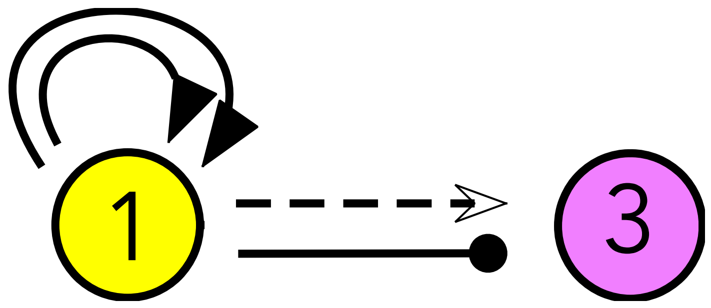
describing the dynamics of the synchronous clusters. [Restriction Theorem]

Identifying $\Delta_{\bowtie}$ with coordinates from a set of representatives of the clusters, the reduced ODEs are precisely the admissible ODEs for the quotient network by $\bowtie$. [Lifting Theorem]

Quotient Network Example







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Application to Bifurcations

We can use balanced colourings to seek solutions of an admissible ODE that have a given synchrony pattern.

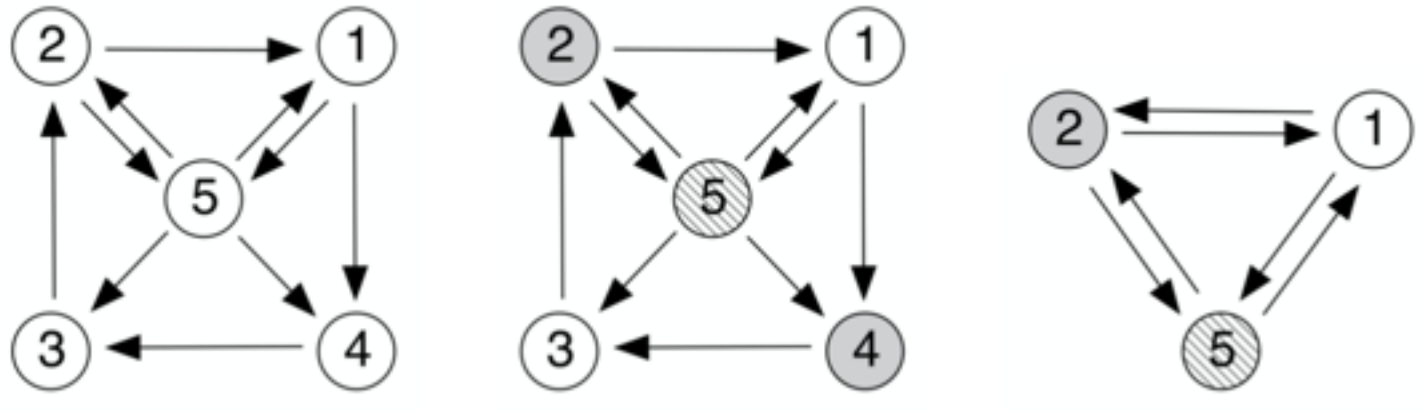
Restrict to the synchrony space, solve there.

This leads to a variety of bifurcation theorems, asserting the existence of such solutions in a 1-parameter family

$$\dot{x}_1 = f(x, \lambda)$$

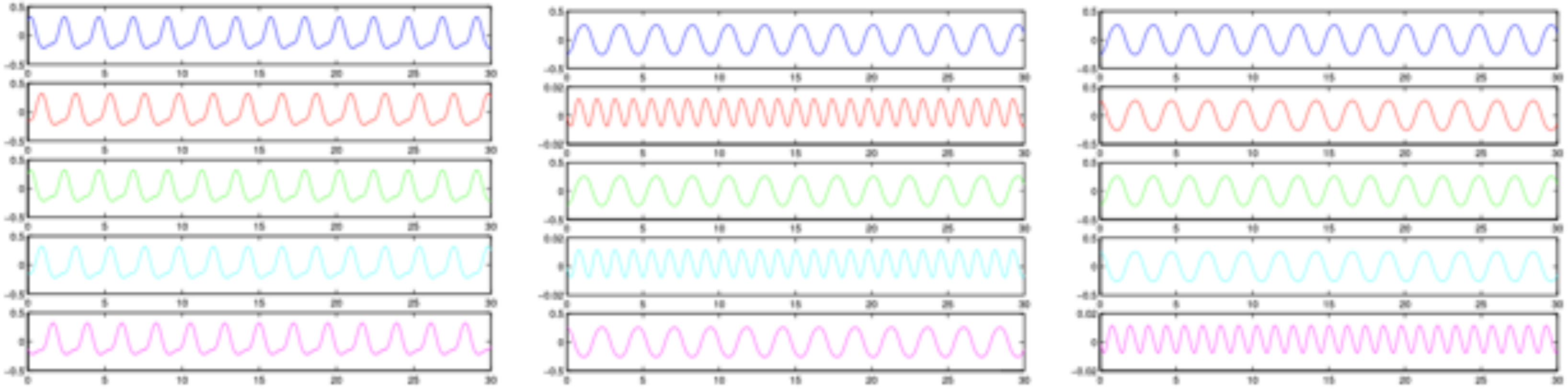
of ODEs, as the parameter λ varies.

Synchronised Periodic Orbits



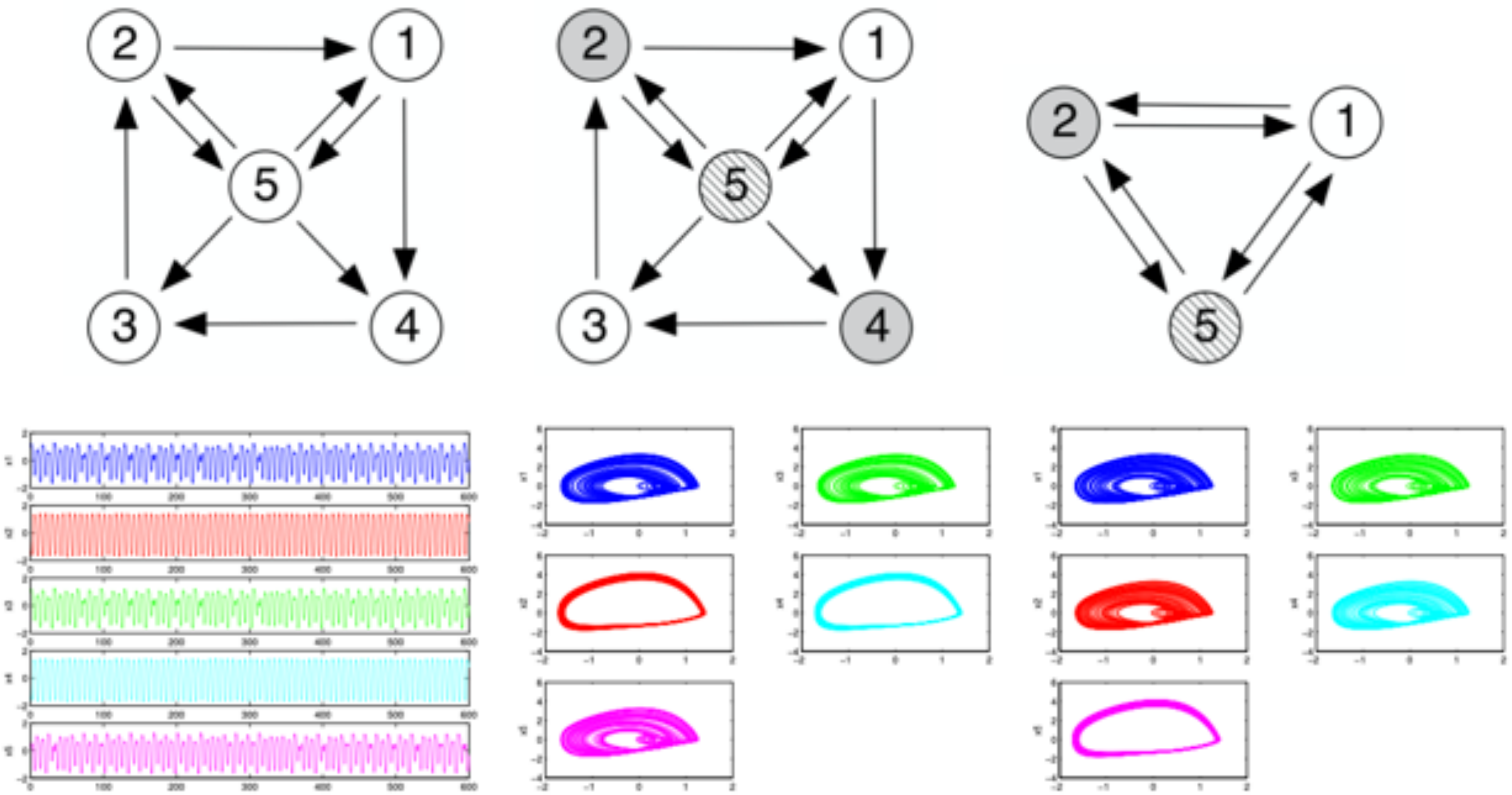
quotient:
3-node,
symmetry
$\mathbf{D}_3 = \mathbf{S}_3$

node 1
node 2
node 3
node 4
node 5



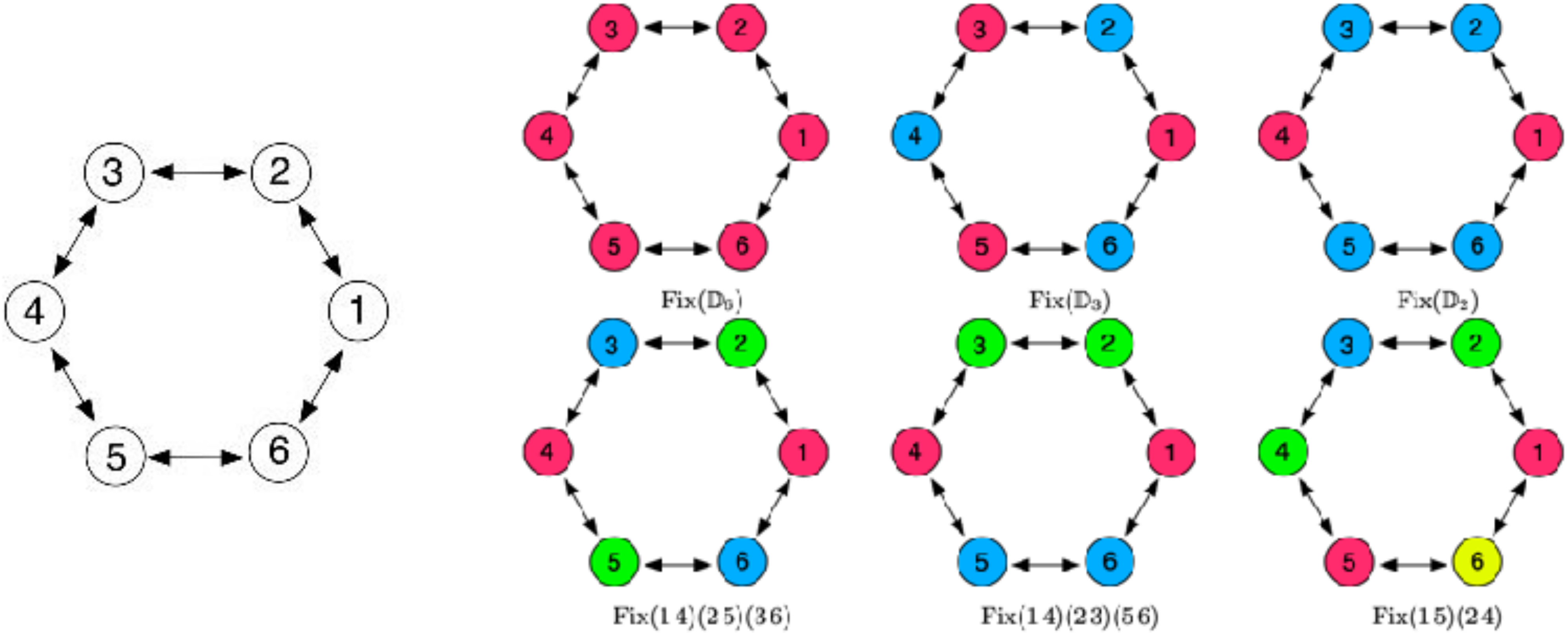
Left: rotating wave. Middle: Double frequency in nodes 2 and 4. Right: Double frequency in node 5.

Synchronised Chaos



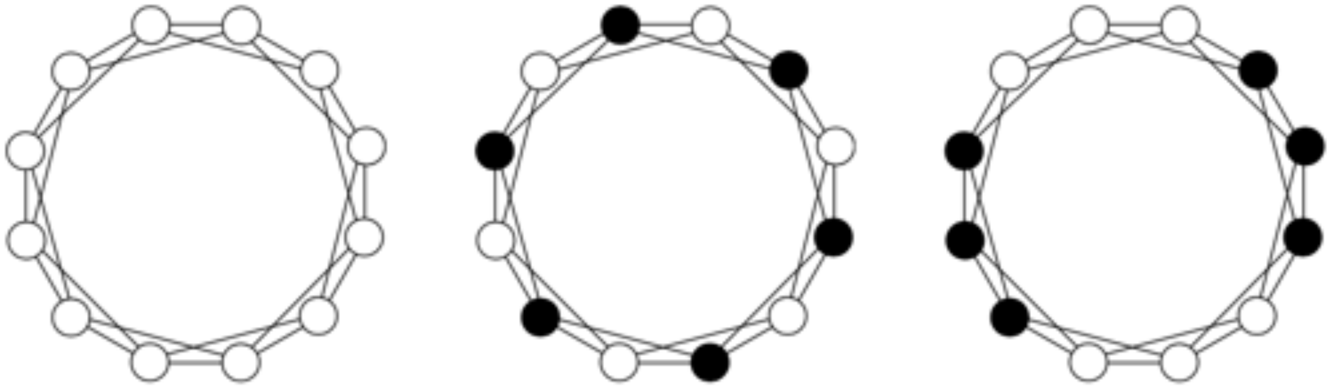
Left: Time series for chaotic attractor with $\mathbf{Z}_2$ symmetry. Middle: Phase plane with nodes 1,3,5 and nodes 2,4 exhibiting symmetry on average. Right: Double frequency in node 5.

Orbit Colourings



Synchrony patterns in a bidirectional ring of six nodes. Colors indicate synchronous clusters. The trivial pattern, in which all six nodes have distinct colors, is omitted.

Orbit Colourings and Exotic Colourings

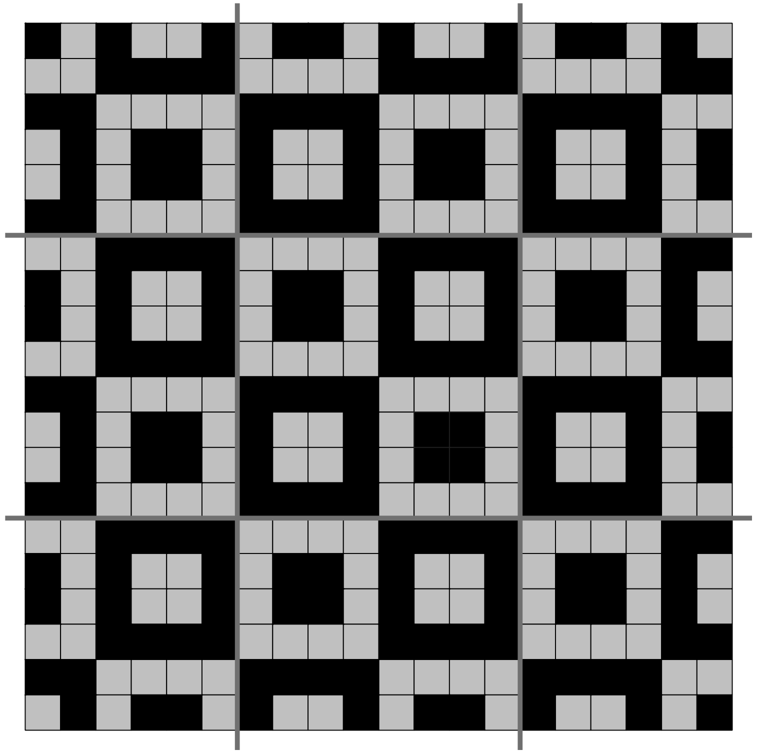


Left: 12-node bidirectional ring with NN and NNN coupling, assumed identical.
Middle: Orbit coloring by $\mathbf{Z}_6$.
Right: Balanced coloring that is not an orbit coloring.

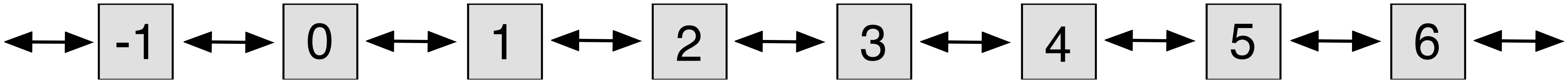
Orbit Colourings and Exotic Colourings

Exotic 2-colour pattern in NN lattice, either $\mathbf{Z}^2$ or its mod-8k reduction (any k).

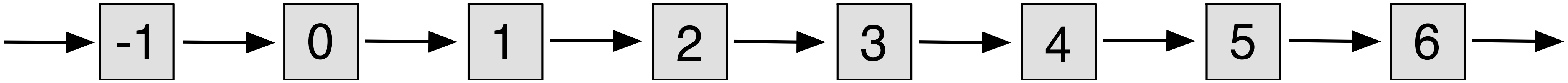
This colouring remains balanced for longer-range connections. Arrow-type given by group orbits under $\mathbf{Z}^2$ semidirect product with $\mathbf{D}_4$.



Linear Lattices



Bidirectional



Unidirectional

Linear Lattices

Theorem 26.4. *Let $k \geq 2$. Every balanced k-coloring on the bidirectional linear lattice with NN connections is obtained by repeating one of the following sequences of colors periodically. (Omit zero and negative terms $(k-1), (k-2)$ for $k \leq 3$. When $k = 2$ types* (a) *and* (b) *are the same.)*

(a) $1\ 2\ \ldots (k-2)\ (k-1)\ k$ [period k]

(b) $1\ 2\ \ldots (k-2)\ (k-1)\ k\ (k-1)\ (k-2)\ \ldots 2$ [period $2k-2$]

(c) $1\ 2\ \ldots (k-2)\ (k-1)\ k\ k\ (k-1)\ (k-2)\ \ldots 2$ [period $2k-1$]

(d) $1\ 1\ 2\ \ldots (k-2)\ (k-1)\ k\ k\ (k-1)\ (k-2)\ \ldots 2$ [period $2k$]

Linear Lattices

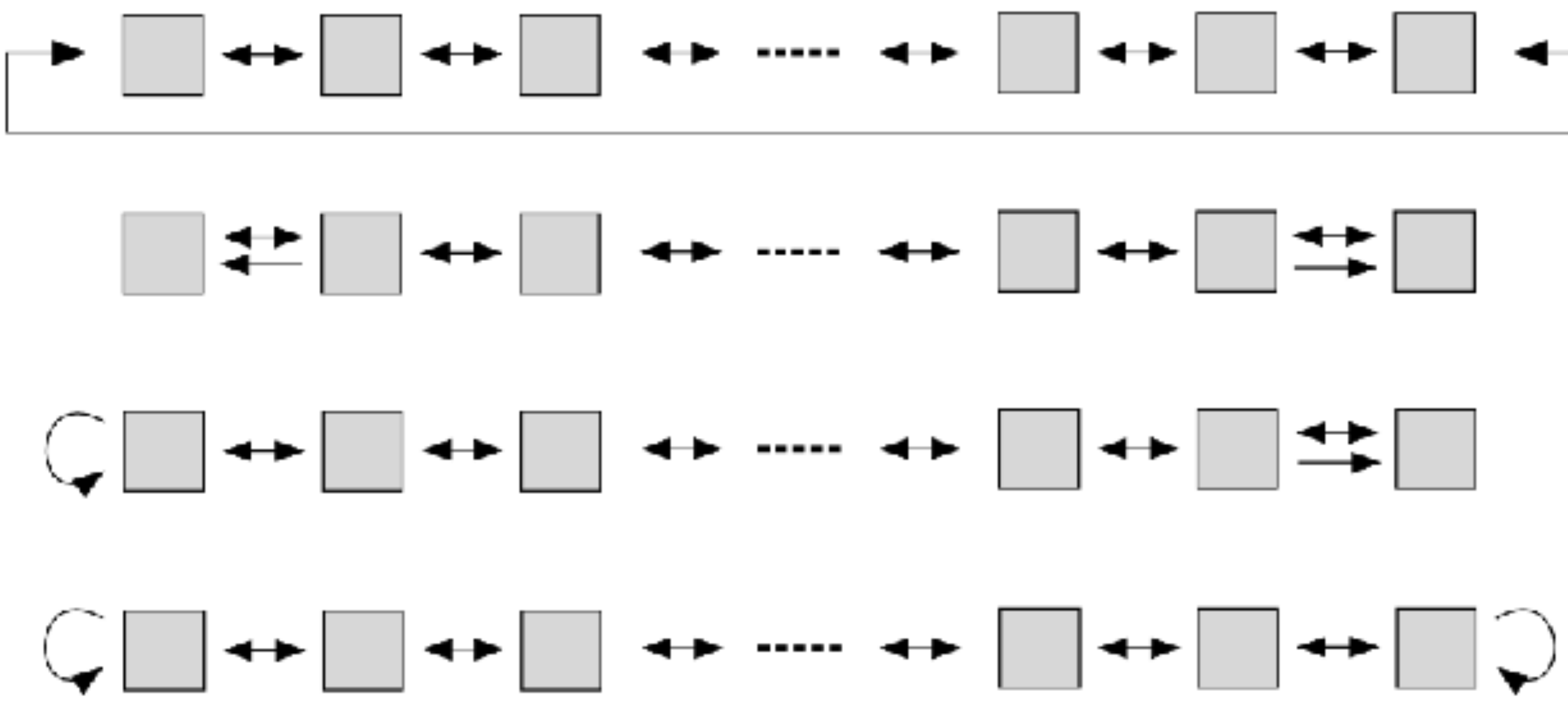


Figure 26.5: The four possible quotient networks for balanced k-colorings of $\mathbb{Z}$. In each network, all nodes have distinct colors $1, 2, \ldots, k$.

'Random' Patterns

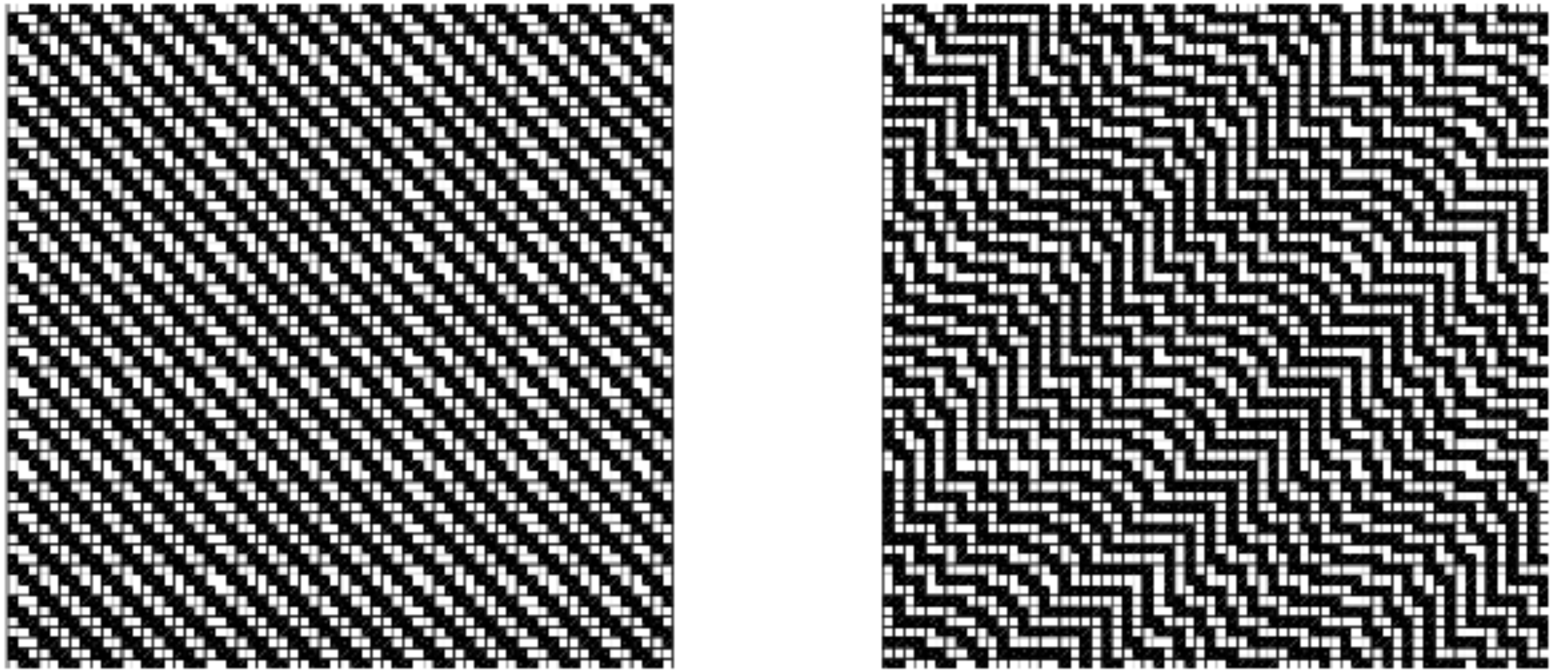


Figure 27.1: Synchrony subspaces of a 2-color 64×64 periodic array. *Left*: The regular pattern. *Right*: Interchanges on a random selection of 25 diagonals.

Square Lattice

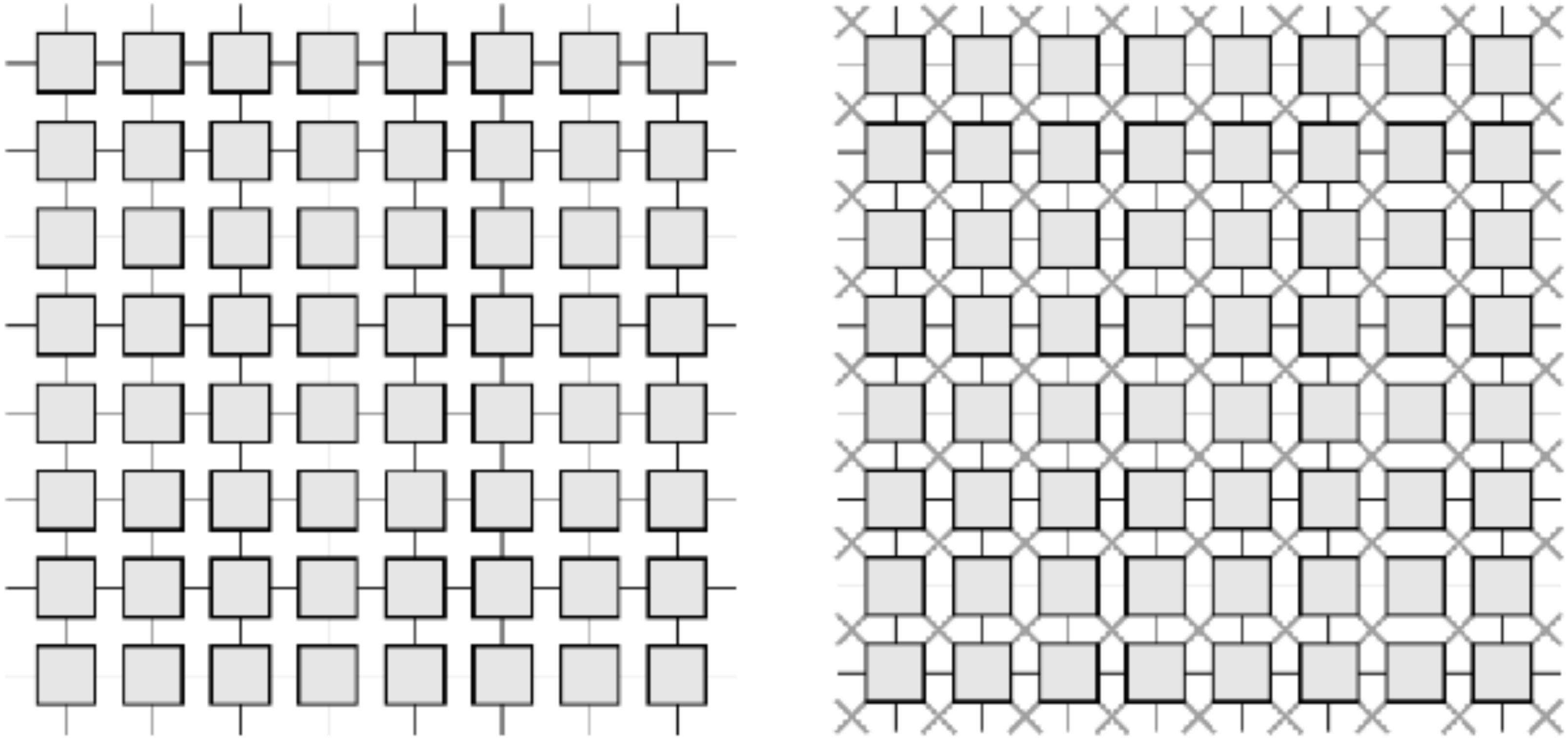


Figure 27.5: *Left*: Square lattice with NN coupling. Squares show nodes. Line segments indicate bidirectional coupling; all couplings identical. *Right*: Gray diagonal lines indicate NNN couplings.

Hexagonal Lattice

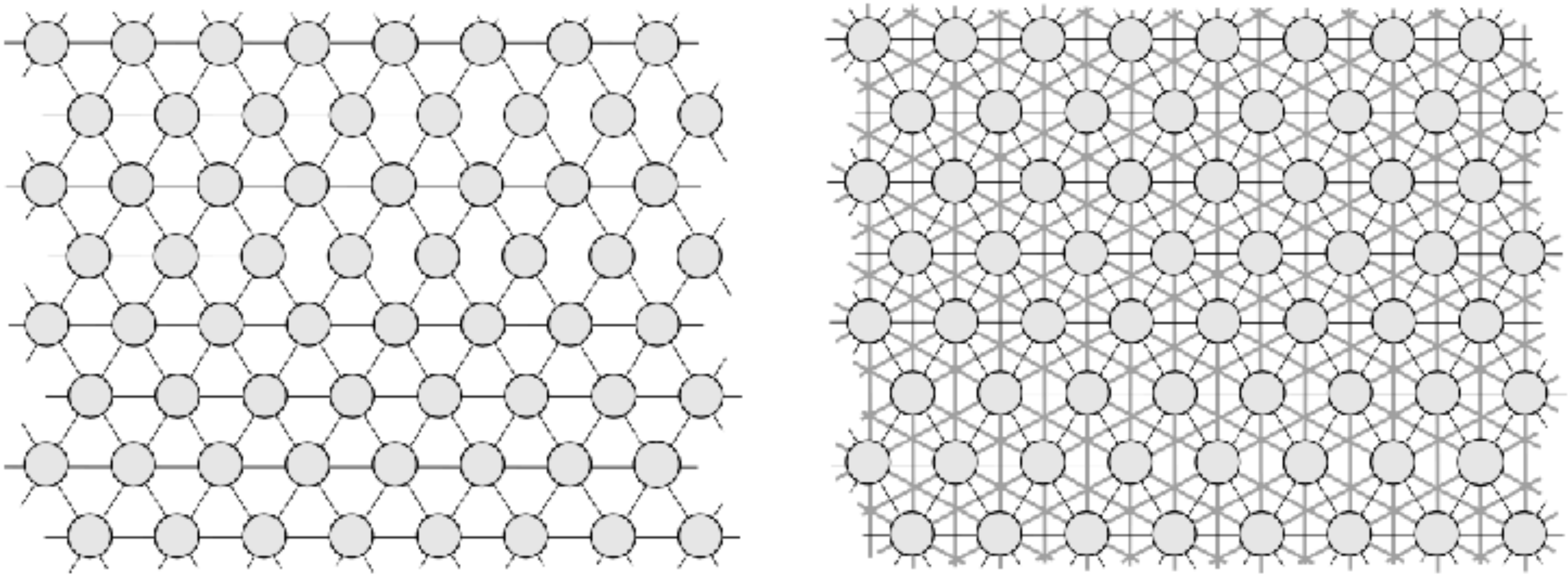
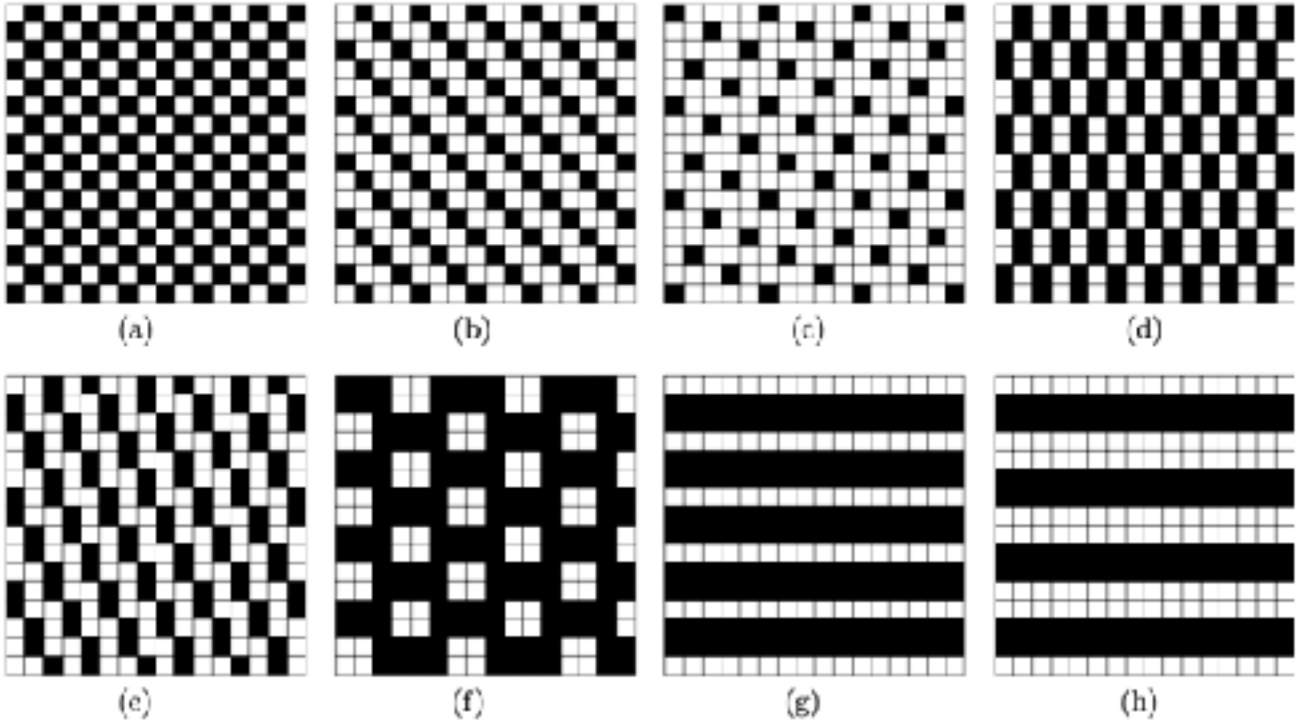
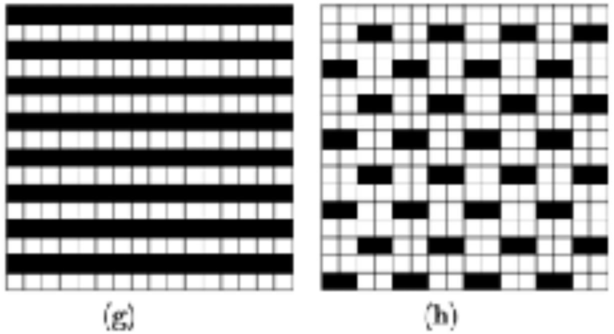


Figure 27.6: *Left*: Hexagonal lattice with NN coupling. Circles show nodes. Line segments indicate bidirectional coupling; all couplings identical. *Right*: Gray diagonal lines indicate NNN couplings.

Square Lattice — Balanced Colourings (Yunjiao Wang & Marty Golubitsky)

Theorem 27.3. *For a square lattice with NN coupling, the balanced 2-colorings, up to symmetry, are the eight doubly periodic patterns in Figure 27.9, together with two infinite families generated from Figure 27.10 by interchanging black and white on diagonals along which black and white nodes alternate.*





Doubly Periodic Quotients – Square Lattice

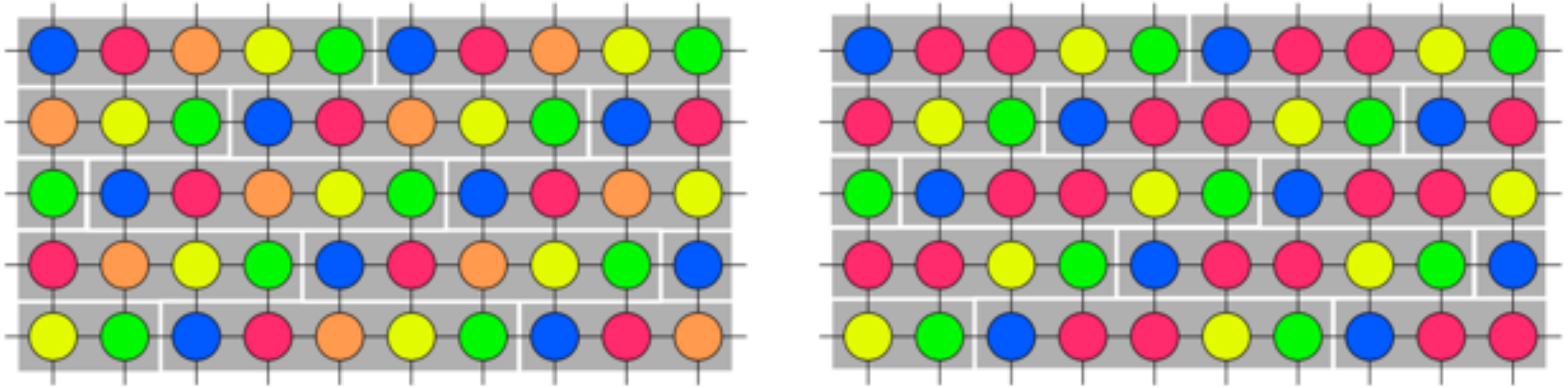


Figure 28.1: *Left*: Balanced 5-coloring of $\mathbb{Z}^2$ with NN coupling. *Right*: Balanced 4-coloring of $\mathbb{Z}^2$ with NN coupling obtained by merging red and orange colors.

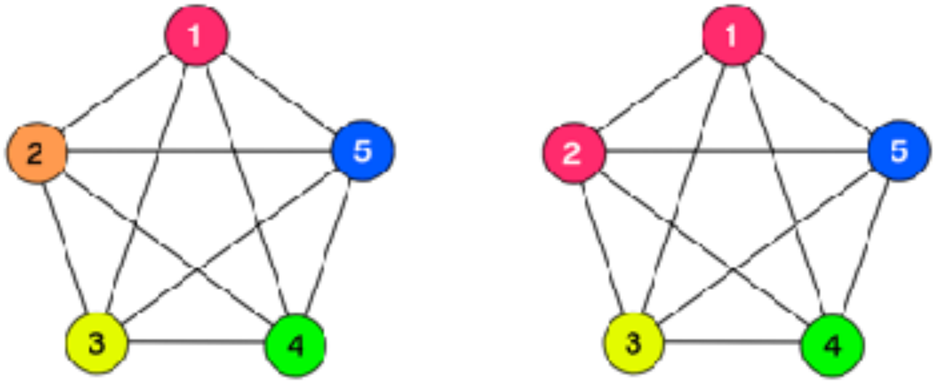


Figure 28.2: *Left*: Quotient $\mathbb{Z}^2/\bowtie$ of $\mathbb{Z}^2$ by the 5-coloring has $\mathbb{S}_5$ symmetry and is all-to-all connected. *Right*: Any coloring on $\mathbb{Z}^2/\bowtie$, such as this one, is balanced, so lifts to a balanced coloring of $\mathbb{Z}^2$ with the same lattice periodicity.

Doubly Periodic Quotients – Square Lattice

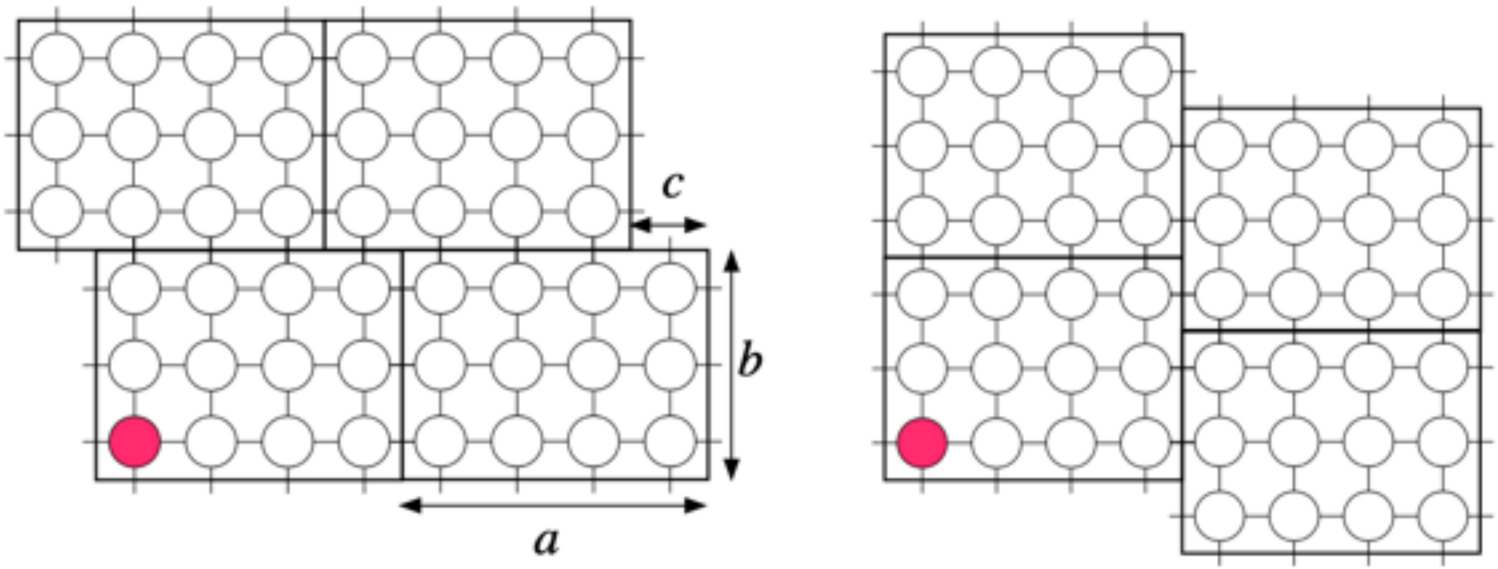


Figure 28.13: Staircase tilings by rectangles. Black disc is at the origin. *Left*: H-tiling, with tiles aligned in horizontal rows, showing parameters (a, b, c). *Right*: V-tiling, with tiles aligned in vertical columns. Thin lines between circles indicate bidirectional arrows.

Doubly Periodic Quotients – Square Lattice

reference	(a,b,c)	$\mathrm{aut}(\mathcal{G})$	order	comments
(a)	$(5,1,2)$	$\mathbb{S}_5$	120	symmetric group of degree 5
(b)	$(6,1,2)$	$\mathbb{O} \times \mathbb{Z}_2^-$	48	full octahedral group $\cong \mathbb{S}_4 \times \mathbb{Z}_2$
(c)	$(8,1,3))$	$\mathbb{S}_4 \wr \mathbb{Z}_2$	1152	exceptional member of family (f)
(d)	$(10,1,3)$	$\mathbb{S}_5 \times \mathbb{Z}_2$	240	not wreath product
(e)	$(4,4,0)$	$\mathbb{Z}_2 \wr \mathbb{S}_4$	384	hyperoctahedral group B_4
(f)	$(a,1,c)$	$\mathbb{Z}_2 \wr \mathbb{D}_{a/2}$	$a2^{a/2}$	$a \geq 6$ even, $c = \frac{a}{2} - 1$
(g)	$(a,2,2)$	$\mathbb{Z}_2 \wr \mathbb{D}_a$	$a2^{a+1}$	$a \geq 4$ and even

Table 28.1: Classification of rank-2 sublattices of $\mathcal{L} = \mathbb{Z}^2$ whose quotient networks have exotic automorphisms.

Exotic Periodic Pattern

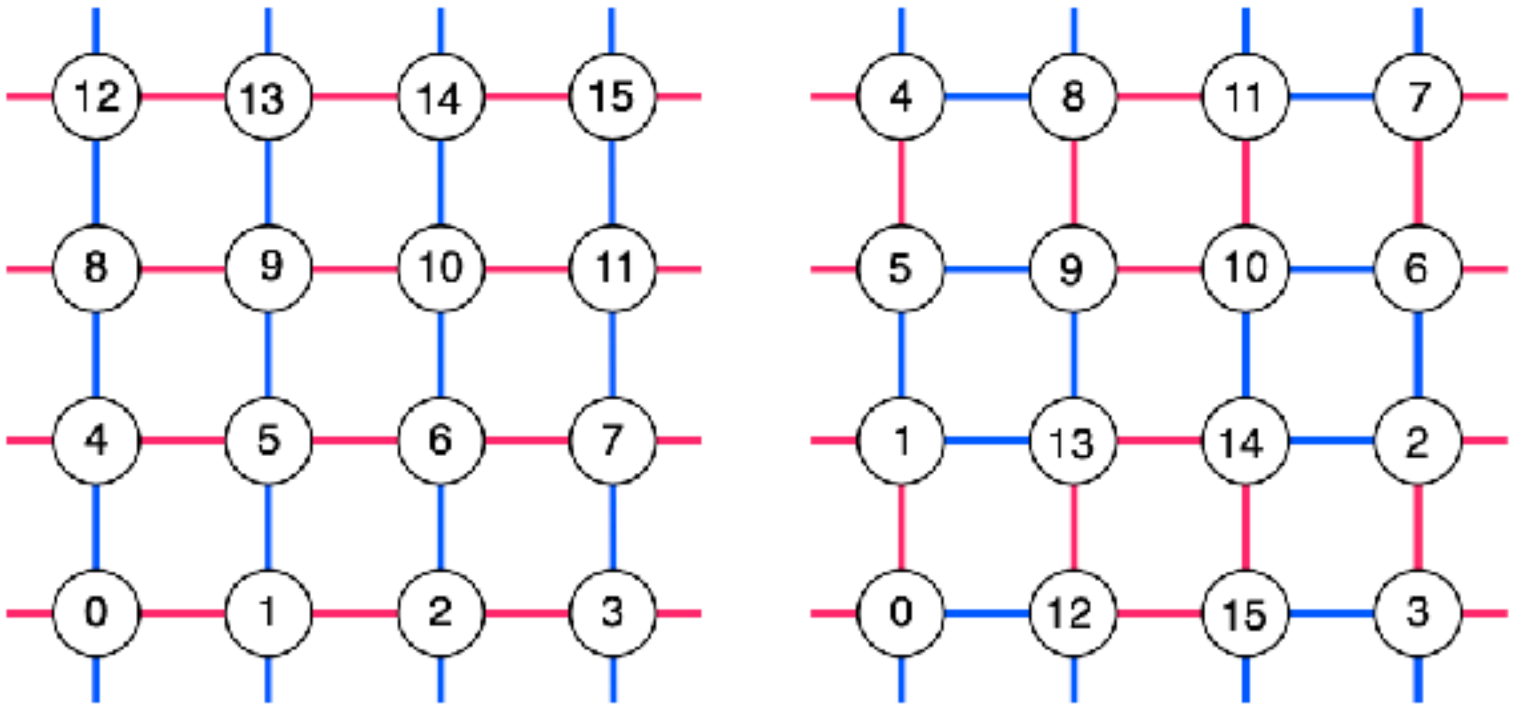


Figure 28.16: Case $b = 4$, $a = 4$. *Left*: Horizontal 4-cycles shown in red, vertical 4-cycles in blue. The other 4-cycles are standard. *Right*: How the nodes and the red and blue 4-cycles transform under α.

Square NN Lattice – Hyperoctahedral Group

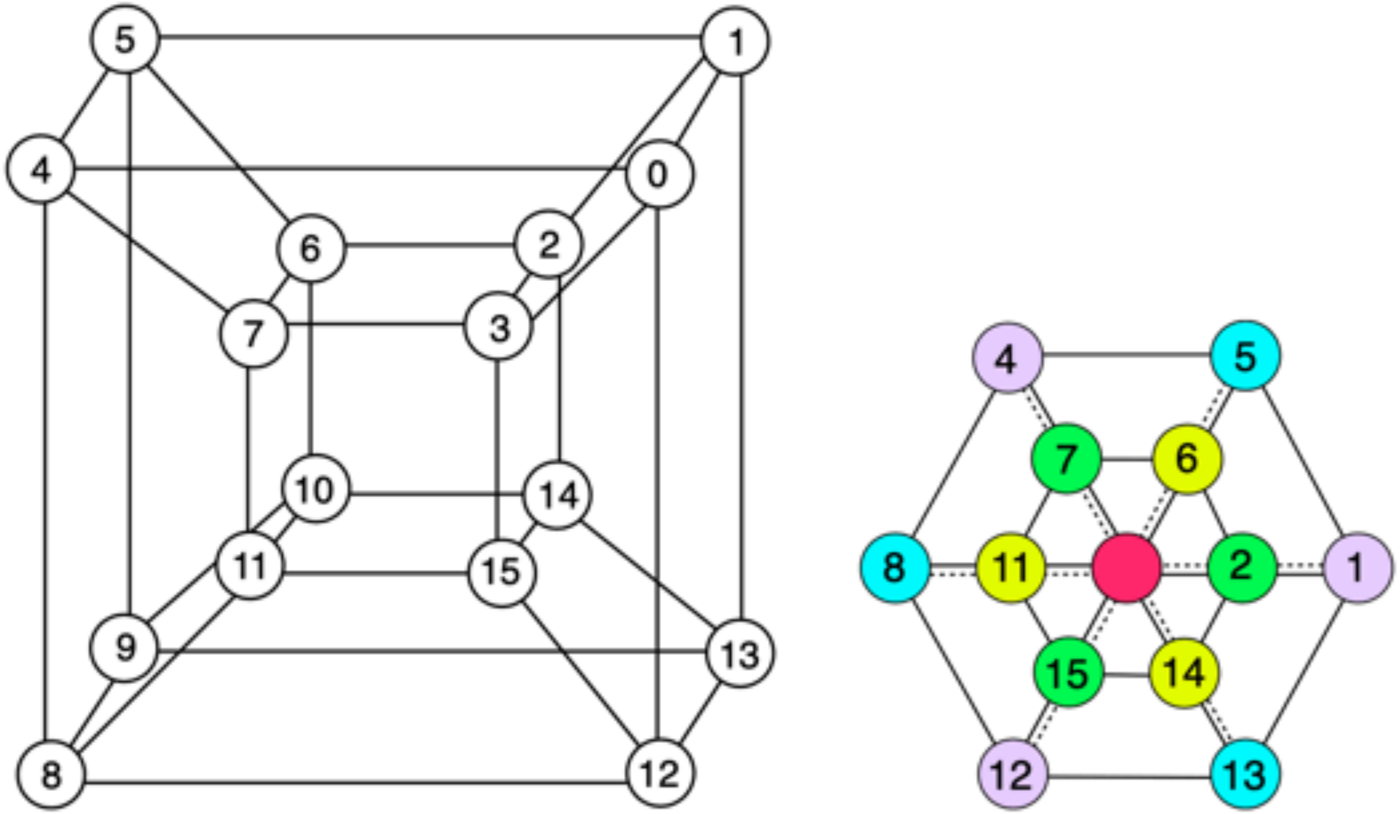


Figure 28.17: *Left*: Isomorphism between the quotient network for $(4,4,0)$ and the edge-graph of the hypercube. *Right*: Quotient network viewed along a main diagonal to illustrate 120° rotational symmetry. Nodes $0,3,9,10$ coincide at the central red node. Dotted edges connect outer nodes to one of these central nodes. Each solid edge connecting to the central node joins to exactly one of nodes $0,3,9,10$ according to the left-hand figure. Colors indicate the component 3-cycles of α.

Opinion Networks

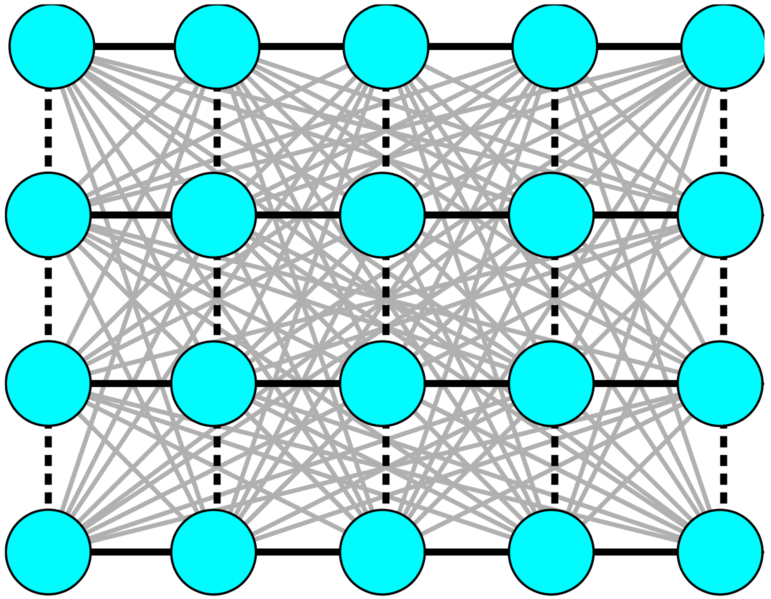
Some models of opinion making are based on a network whose nodes form an m x n array. Arrows come in four types:

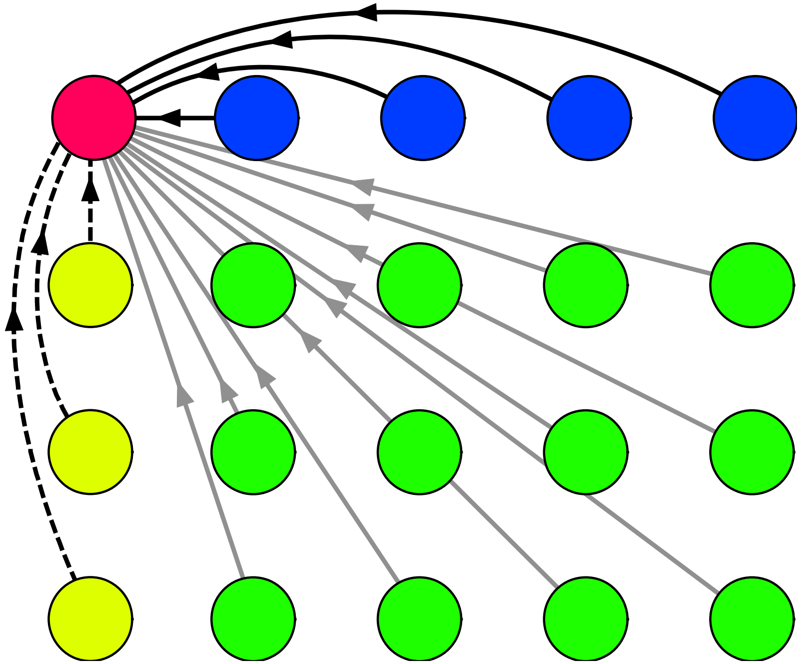
Node arrows — 'internal' node symbol.

Row arrows — all-to-all connected in each row, with identical arrow types.

Column arrows — ditto for columns.

Diagonal arrows — connect nodes in different rows and different columns.





Opinion Networks

Columns correspond to options j for various choices.

Rows correspond to agents i who assign preferences x_{ij} to option j.

These networks have symmetry group $\mathbf{S}_m \times \mathbf{S}_n$.

	issue 1	issue 2	issue 3	issue 4
voter 1	3	9	8	1
voter 2	9	3	2	7
voter 3	5	5	5	5

Opinion Networks

Consider bifurcations from **consensus** states to **dissensus** ones.

According to equivariant dynamics, these are governed by the **irreducible representations** of the symmetry group.

Here there are four of these:

V_1 : All entries x_{ij} equal.

V_2 : All rows the same with row-sum 0.

V_3 : All columns the same with column-sum 0.

V_4 : All row-sums and column-sums 0.

Opinion Networks

Look for steady-state bifurcations from states in V_1.

According to equivariant dynamics, generically the kernel of the Jacobian at the bifurcation point is an **absolutely irreducible representation** of the symmetry group. All V_j are absolutely irreducible, and we are looking for bifurcations with kernel V_4.

Warning

With network constraints (admissible ODE) this theorem need not apply.
In this case, however, it does.

(The linear admissible maps are the same as the linear equivariant maps.)

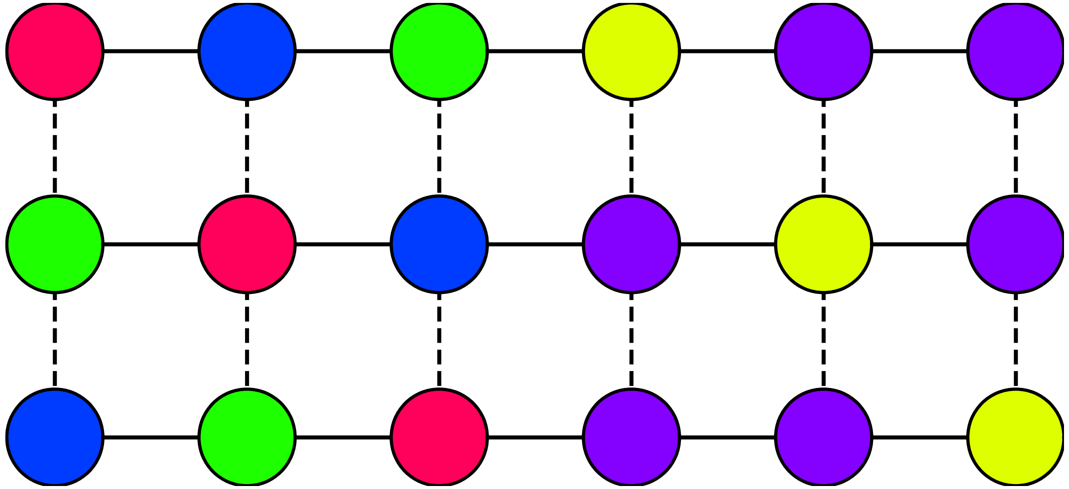
Opinion Networks

Equivariant Branching Lemma: Generically there is a bifurcating branch for each **axial** subgroup H. This means that the fixed-point space of H intersects V4 in a 1-dimensional space.

Disadvantages:

1 This is difficult. Complicated group theory.

2 It omits **exotic** colourings, not given by fixed-point spaces.



Opinion Networks

Instead, we use the network structure.

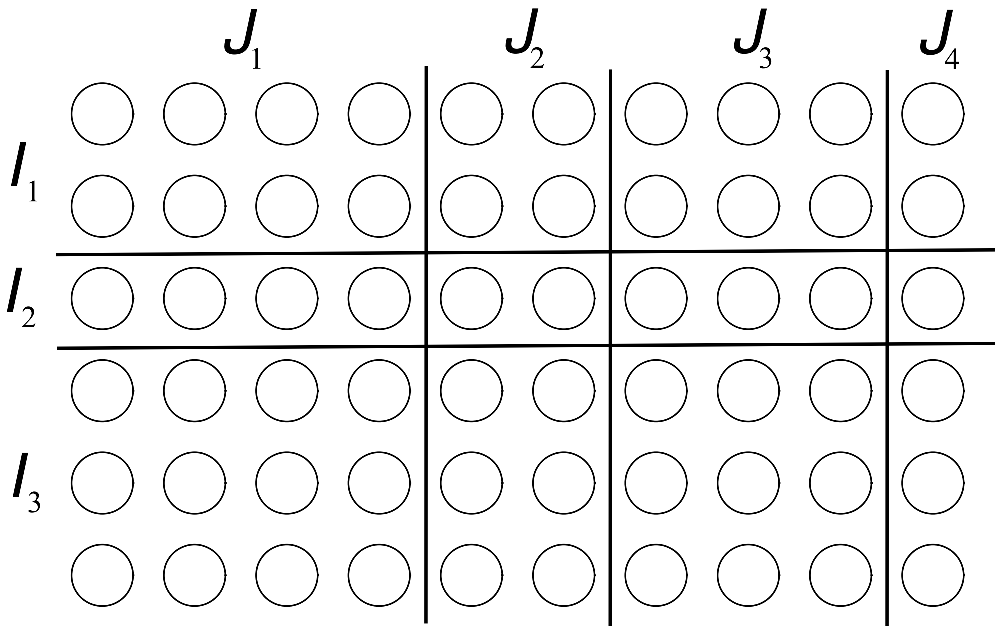
There is an analogue of the Equivariant Branching Lemma, in which axial subgroups are replaced by **axial** balanced colourings. In this case, these are the balanced colourings such that $\triangle_{\bowtie}$ intersects V_4 in a 1-dimensional space.

It is possible to classify all balanced colourings, modulo the classification of **Latin rectangles**.

We can then read off the classification of the axial balanced colourings.

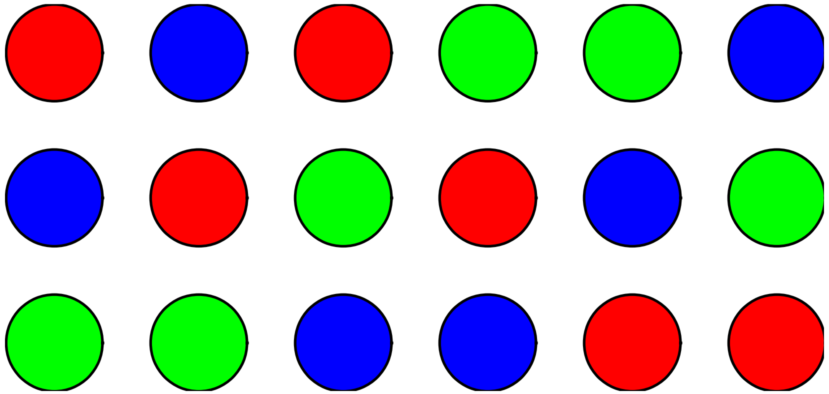
Opinion Networks

A coloring is balanced for G_{mn} provided it is balanced when `diagonal' arrows are deleted and `node' internal arrows are ignored.

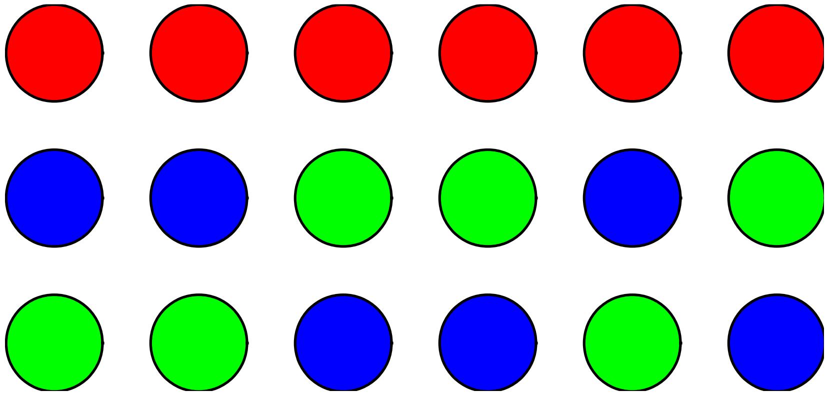


A **Latin rectangle** is an a x b array of coloured nodes, such that:
(a) Each colour appears at least once in every row and every column.
(b) Each row is balanced.
(c) Each column is balanced.

Opinion Networks



Latin rectangle with 3 colors.



This is not a Latin rectangle.
It satisfies (b) and (c) but not (a).

Opinion Networks

order	number
1	1
2	1
3	1
4	4
5	56
6	9,408
7	16,942,080
8	535,281,401,856
9	377,597,570,964,258,816
10	7,580,721,483,160,132,811,489,280
11	5,363,937,773,277,371,298,119,673,540,771,840

Table 1: numbers of reduced Latin squares of order n

Theorem

A colouring of G_{mn} is balanced if and only if it is conjugate under $\mathbf{S}_m \times \mathbf{S}_n$ to a tiling by rectangles, meeting along edges, such that:

(a) Each rectangle is a Latin rectangle.

(b) Distinct rectangles have disjoint sets of colours.

Opinion Networks

Theorem 3.1. *The axial colorings on the dissensus space are as follows:*

(a) $\begin{bmatrix} B_0 & B_1 \end{bmatrix}$ *where* B_0 *is a rectangle with only one color* (Y) *and* B_1 *is a Latin rectangle with two colors* (R, B)*. On state space, all rows and columns of* B_1 *sum to zero. Possibly* B_0 *is empty.*

(b) $\begin{bmatrix} B_0 \\ B_1 \end{bmatrix}$ *where* B_0 *is a rectangle with only one color* (Y) *and* B_1 *is a Latin rectangle with two colors* (R, B)*. On state space, all rows and columns of* B_1 *sum to zero. Possibly* B_0 *is empty; if so, this is the same as (a) with empty* B_0*.*

(c) $\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}$ *where the* B_{ij} *are rectangles with only one color and the four rectangles are color-disjoint. The row- and column-sum conditions then imply three independent linear conditions on the four variables corresponding to the colors*

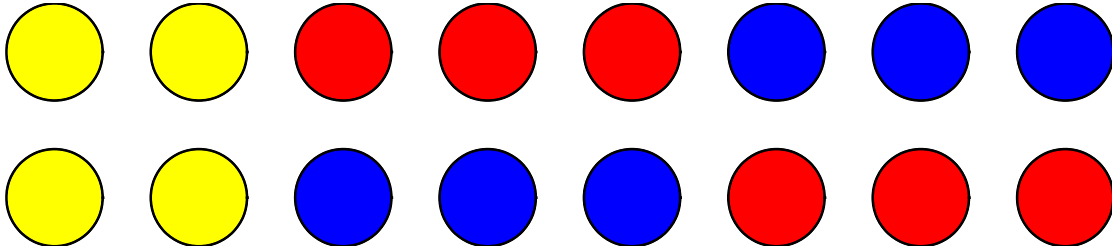
Opinion Networks

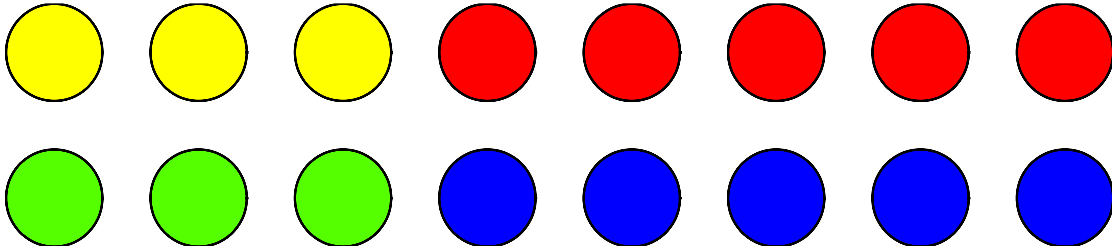
$|I_1| = a$, $|I_2| = b = m - a$, $|J_1| = c$, and $|J_2| = d = n - c$. Here all of $a, b, c, d > 0$. Let the colors be R for B_{11}, B for B_{21}, G for B_{12}, and Y for B_{22}. Then the column- and row-sums imply

$$\begin{aligned} 0 &= ax_R + bx_B \\ 0 &= ax_G + bx_Y \\ 0 &= cx_R + dx_G \\ 0 &= cx_B + dx_Y \end{aligned}$$

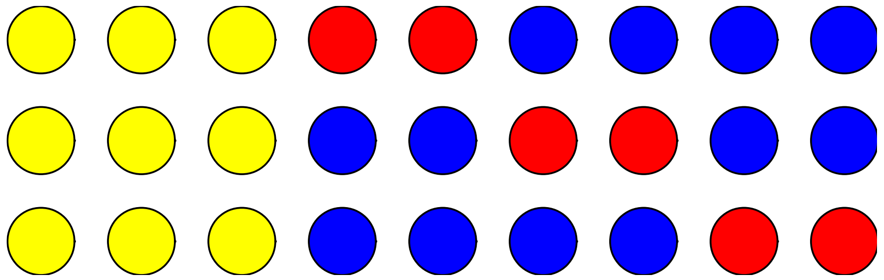
$$\begin{aligned} x_B &= = -\tfrac{a}{b}x_R \\ x_G &= -\tfrac{c}{d}x_R \\ x_Y &= -\tfrac{c}{d}x_B = \tfrac{ac}{bd}x_R \end{aligned}$$

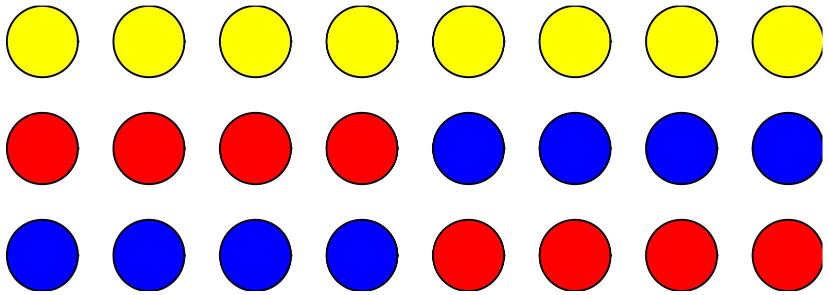
Opinion Networks: 2xn axial colourings

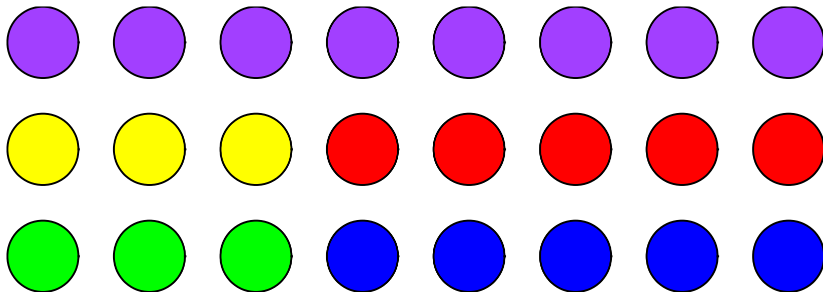




Opinion Networks: 3xn axial colourings

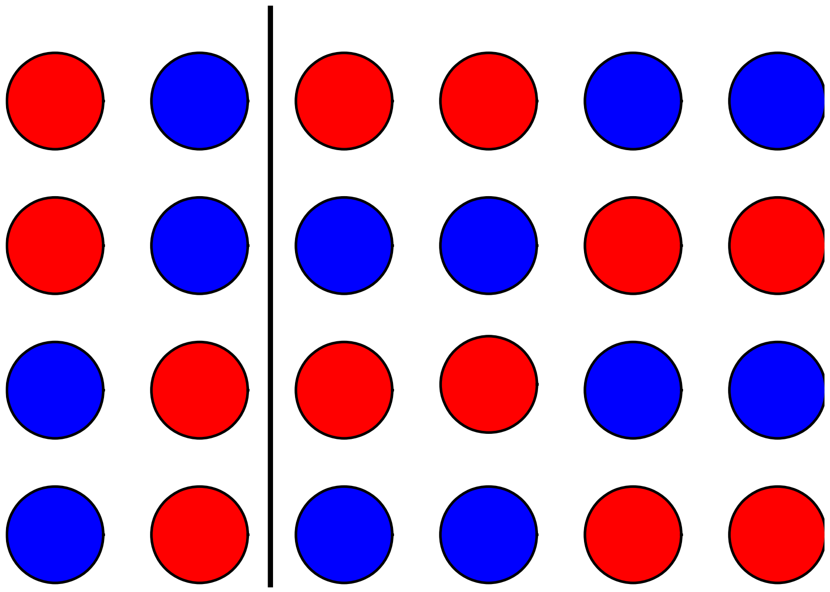






Opinion Networks: 4xn axial colourings

Can be done, but I'll omit it. However, there are (lots of) exotic colourings for n large enough.



Opinion Networks: m x n axial colourings

There is a classification in terms of **minimal** Latin rectangles. A finite set of these generates all possible axial colourings, up to conjugacy.

The minimal Latin rectangles can be computed by efficient algorithms.

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