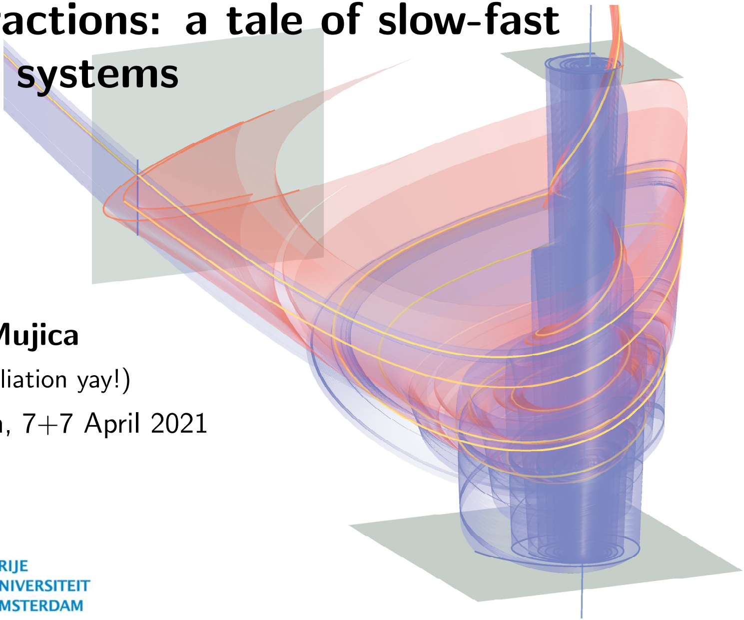
Slow Manifolds, Invariant Manifolds and their interactions: a tale of slow-fast dynamical systems



José Mujica

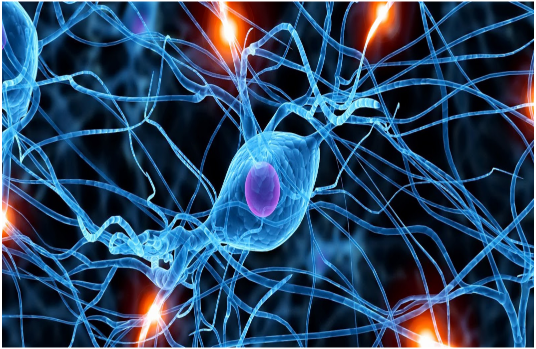
VU (new affiliation yay!)

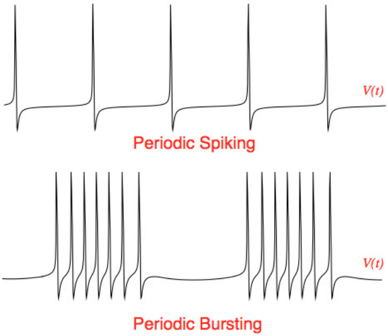
VU Colloquium, 7+7 April 2021

VU VRIJE UNIVERSITEIT AMSTERDAM

General context

– Systems with slow and fast episodes I: Neurons (Spiking, bursting).



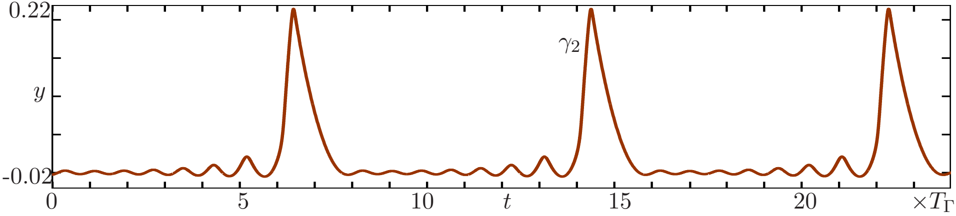


[C. Morris, H. Lecar, *Voltage oscillations in the Barnacle giant muscle fiber*, Biophys. J. 35 (1981), 193–213.]

General context

– Systems with slow and fast episodes II: Chemical reactions.

– Mixed-mode oscillations (MMOs).



– Aim: To study new mechanisms for such dynamics.

Slow-Fast Systems

Slow time-scale (t):

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

$$\Big\Downarrow \varepsilon \to 0$$

$$\begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

$$\overset{t=\varepsilon\tau}{\Longrightarrow}$$

Fast time-scale (τ):

$$\begin{aligned} x' &= f(x,y) \\ y' &= \varepsilon g(x,y) \end{aligned}$$

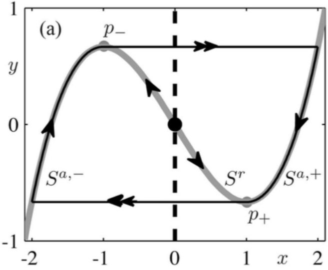
$$\Big\Downarrow \varepsilon \to 0$$

$$\begin{aligned} x' &= f(x,y) \\ y' &= 0 \end{aligned}$$

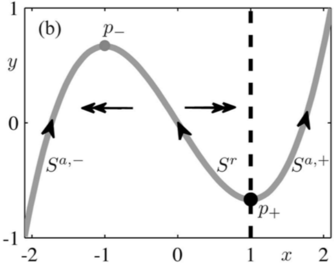
- $0 < \varepsilon \ll 1$.
- $x \in \mathbb{R}^n$: fast variables; $y \in \mathbb{R}^m$: slow variables.
- Solutions have slow and fast segments ($\varepsilon > 0$ small).
- Singular limit ($\varepsilon \to 0$): slow subsystem and fast fibers.
- Critical manifold: $\{f = 0\} = S^a \cup S^r \cup S^s \cup F$.
- Singularly perturbed system ($\varepsilon \neq 0$).

2D example: Van der Pol relaxation-oscillator

$$\varepsilon\dot{x} = y - \tfrac{1}{3}x^3 + x \qquad \text{(fast)}$$
$$\dot{y} = \lambda - x \qquad \text{(slow)}$$



$\lambda = 0$



$\lambda = 1$

(Hopf Bif.)

[Figures: Mixed-mode oscillations with multiple-time-scales, SIAM review (2012)]

2D example: Van der Pol relaxation-oscillator

$$\varepsilon\dot{x} = y - \frac{1}{3}x^3 + x \qquad \text{(fast)}$$
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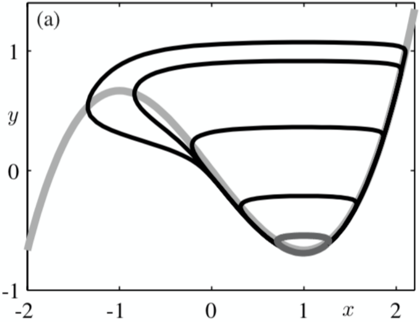
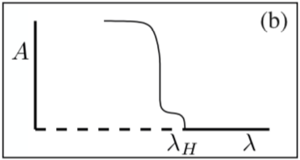


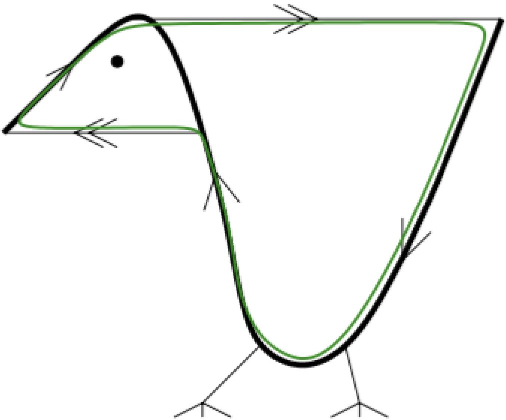
Canard explosion



Singular Hopf bifurcation

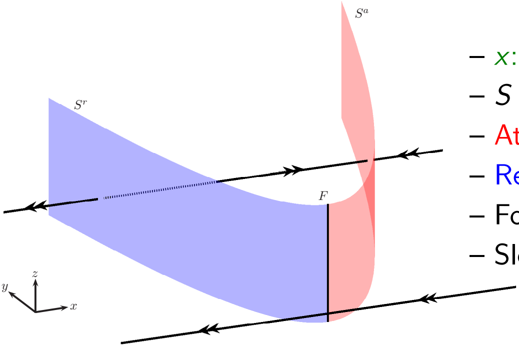
[Figures: Mixed-mode oscillations with multiple-time-scales, SIAM review (2012).]

A canard...



[Figure: M. Wechselberger, Existence and bifurcation of Canards in $\mathbb{R}^3$ in the case of a Folded Node, SIADS (2005).]

$\mathbb{R}^3$: One fast, two slow

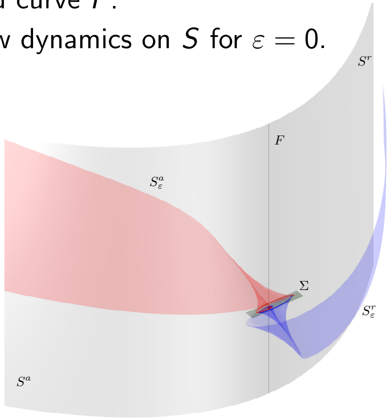


Critical Manifold

- x: fast; y, z: slow.
- $S = \{f(x, y, z) = 0\} = S^a \cup S^r \cup F$.
- Attracting sheet S^a.
- Repelling sheet S^r.
- Fold curve F.
- Slow dynamics on S for $\varepsilon = 0$.

Slow Manifold

- GSPT: For $\varepsilon > 0$, $S^a \approx S^a_\varepsilon$ and $S^r \approx S^r_\varepsilon$.
- Away from folds (normally hyperbolic).
- Solutions remain slow on S^a_ε and S^r_ε.
- Source of interesting dynamics.



Picture: Slow manifolds in the Koper Model

[**J. M.**, B. Krauskopf, H.M. Osinga, JCD, 4(1) (2017)]

Folds and folded singularities

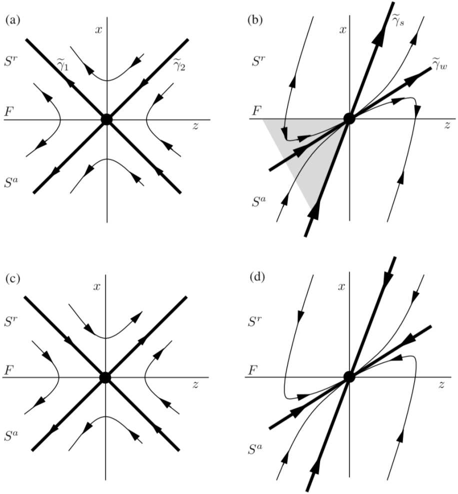
$$\begin{aligned} \varepsilon\dot{x} &= f(x,y,z) \\ \dot{y} &= g(x,y,z) \\ \dot{z} &= h(x,y,z) \end{aligned}$$

- p^* is a fold point (generic):

$$f(p^*) = 0, \quad \frac{\partial f}{\partial x}(p^*) = 0,$$

$$\frac{\partial^2 f}{\partial x^2}(p^*) \neq 0, \quad D_{(y,z)} f(p^*) \quad \text{has full rank one}$$

- The slow flow is not well defined ...before desingularization.
- Folded singularities (singularities of the desingularized flow).
- Not (necessarilly) equilibria.

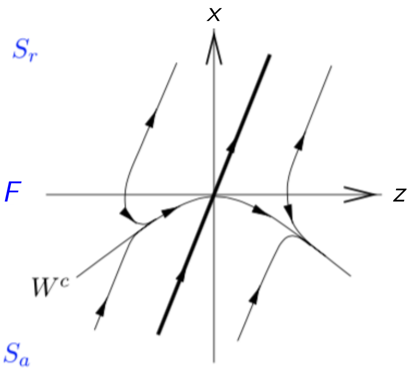


Folded saddle

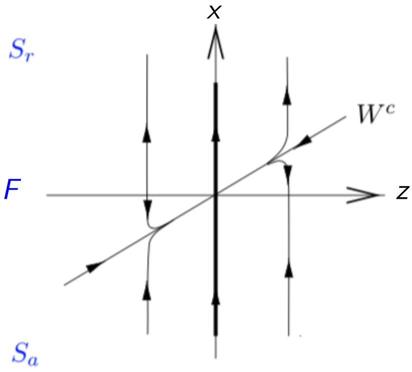
Folded node (canards)

[Figure: Mixed-mode oscillations with multiple-time-scales, SIAM review (2012)]

Folded saddle-node (of type II)



folded saddle-node (type I)



folded saddle-node (type II)

–Folded singularities can undergo bifurcations.

–FSNII involves folded singularity and equilibrium point.

–Fact: FSNII is $O(\varepsilon)$ from Singular Hopf bifurcation.

[Figure: M. Wechselberger, Local analysis near a folded saddle-node singularity, J. Diff. Equations (2010).]

Model: Singular Hopf bifurcation

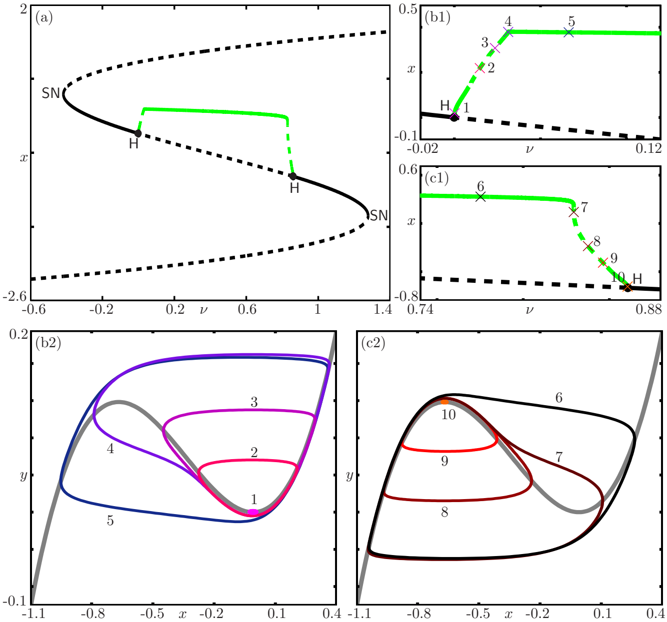
$$
\begin{aligned}
\varepsilon \dot{x} &= f(x,y,z) &= y - x^2 - x^3 \\
\dot{y} &= g(x,y,z) &= z - x \\
\dot{z} &= h(x,y,z) &= -\nu - ax - by - cz
\end{aligned}
$$

Why???

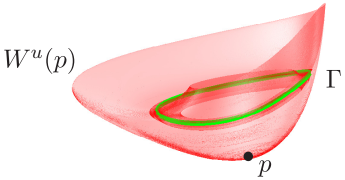
- For $0 < \varepsilon \ll 1$: slow-fast system.
- x: fast variable; y, z: slow variables.
- 'Normal form' introduced by Guckenheimer *.
- Generic/typical.
- Equilibrium p undergoes a Singular Hopf Bifurcation.
- Contains the Koper Model: Shilnikov homoclinic scenario.
- Novel and different scenario for MMOs.
- Folded singularity and equilibrium.
- Slow manifolds and invariant manifolds involved.
- Mix between classical ingredients and slow-fast ingredients.

*[J. Guckenheimer, P. Meerkamp, SIAM J. Appl. Dyn. Syst., 11 (2012), pp. 1325–1359.]

The basics: bifurcation diagram



Classical ingredients



– p: Saddle-focus equilibrium.

– $W^u(p)$: Unstable manifold of p.

$\{q \in \mathbb{R}^3 : \varphi_q(t) \to p, \text{ when } t \to -\infty\}$

– Γ: Attracting periodic orbit.

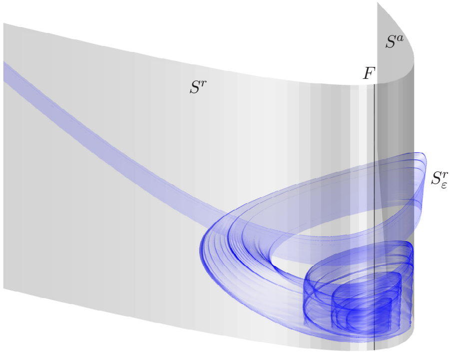
(Boundary of $W^u(p)$)

Slow-fast ingredients

– Slow manifolds: Surfaces of solutions that remain slow for an $O(1)$-time on the slow-time scale.

(Fenichel theory, GSPT)

– S^r_ε: Repelling slow manifold.



DYNAMICS?

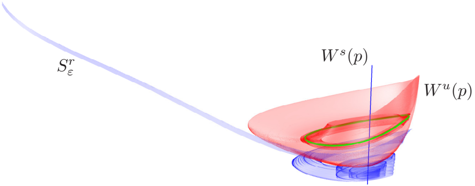
Tool: **BVP setup + continuation (AUTO)**

Interlude I: (Numerical) Continuation

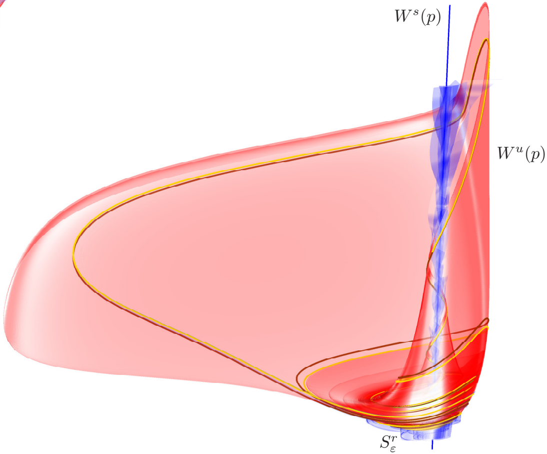
$$\dot{x} = f(x, \lambda), \quad x \mapsto f(x, \lambda)$$
$$x \in \mathbb{R}^n(\mathbb{C}^n), \lambda \in \mathbb{R}^p, f \in \mathbb{C}^r, r \geq 1$$

- Basic idea: to compute an implicitly defined curve that defines a dynamical object.
- Find position of equilibria/fixed points: $x^* = x^*(\lambda)$.
- Determine stability of equilibria/fixed points: $x^* = x^*(\lambda)$.
- Find position and stability of periodic solutions.
- Detect (local and global) bifurcations as λ is varied.
- Calculate and visualize higher-dimesional global invariant sets and other relevant objects.
- Ultimate goal: to understand and describe the overall dynamics in phase space.
- Numerical methods: Newton's method, Pseudo-arclength continuation, boundary-value problems.
- Software packages: AUTO, MATCONT, CONTENT, COCO, etc.

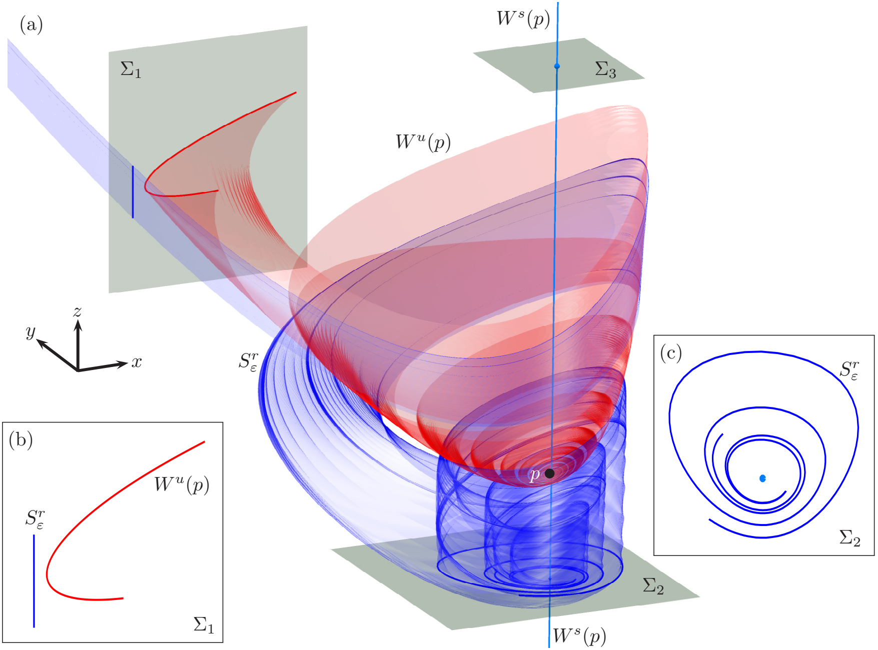
Interactions between $W^u(p)$ and S^r_ε



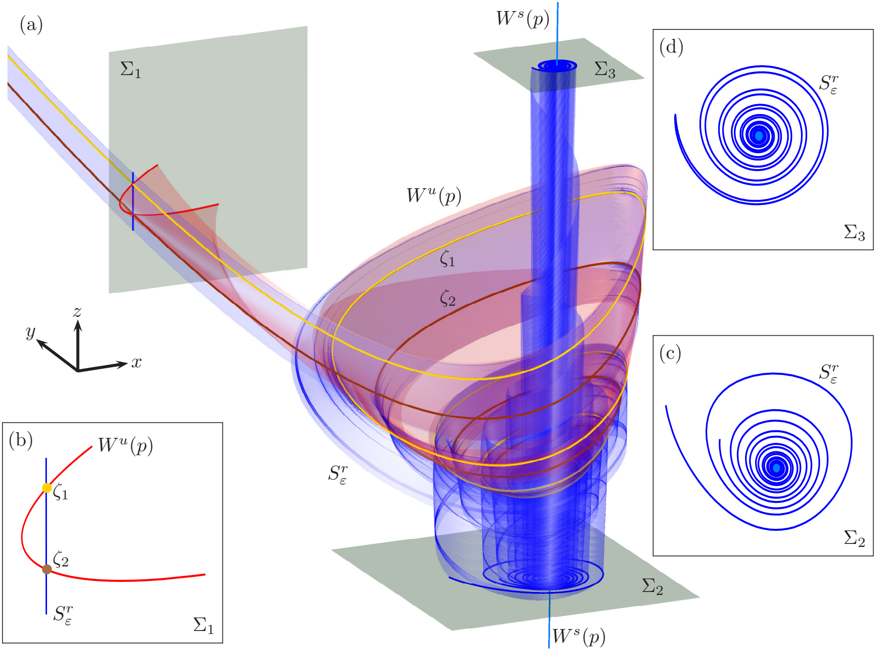
Transition through quadratic tangency



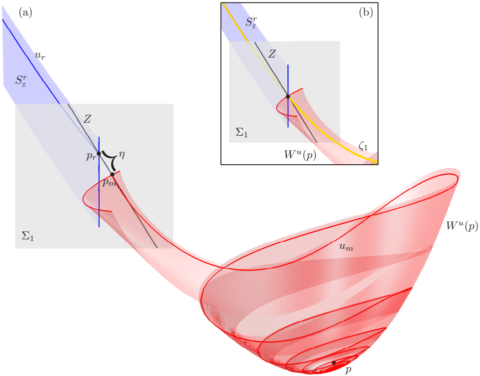
Local picture before tangency

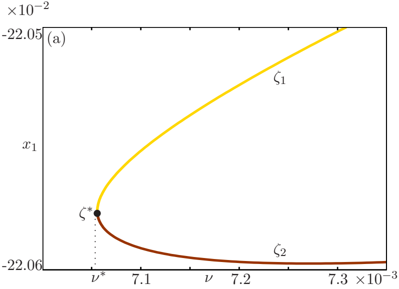


Local picture after tangency



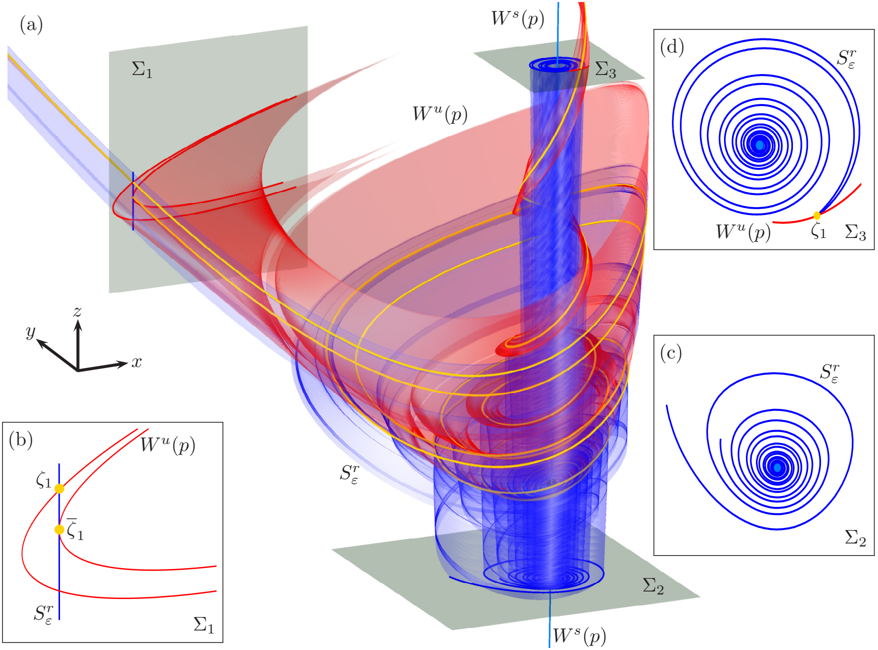
Interlude II: Finding connecting canard orbits and tangency



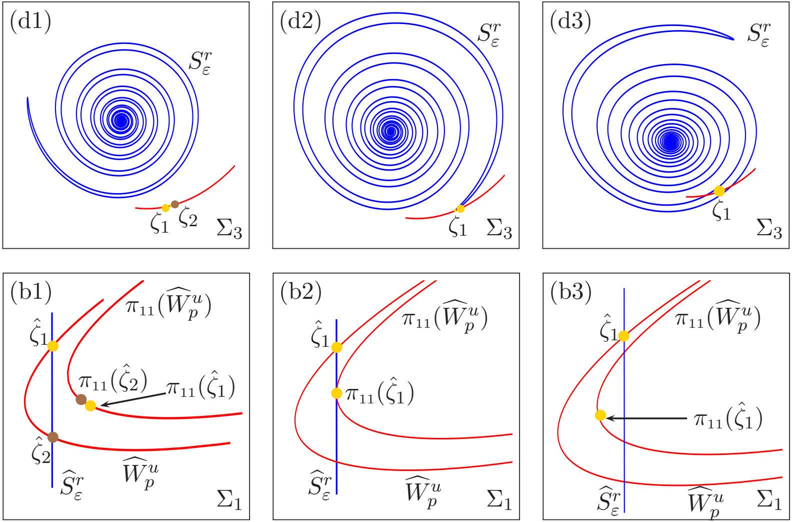


- Lin's method approach; also implemented for the systematic detection of canard orbits near folded nodes in other 3D models.
- Tangency as a fold.

Consequence: Secondary tangency

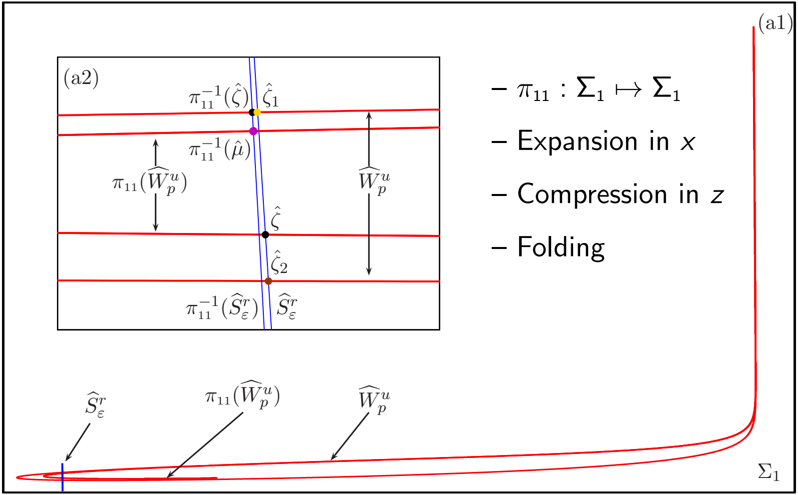


Secondary intersections



$\nu = 0.00712$ $\nu = \nu^{**}$ $\nu = 0.00722$

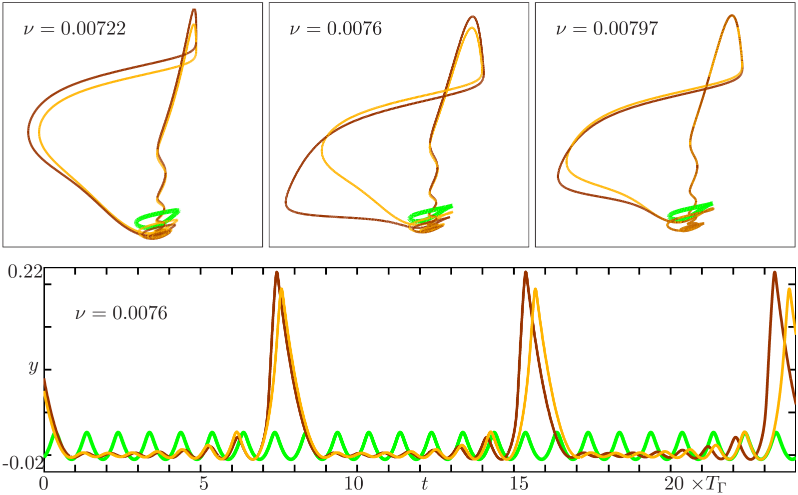
Global intersection in Σ_1



Horseshoe structure implies periodic orbits!

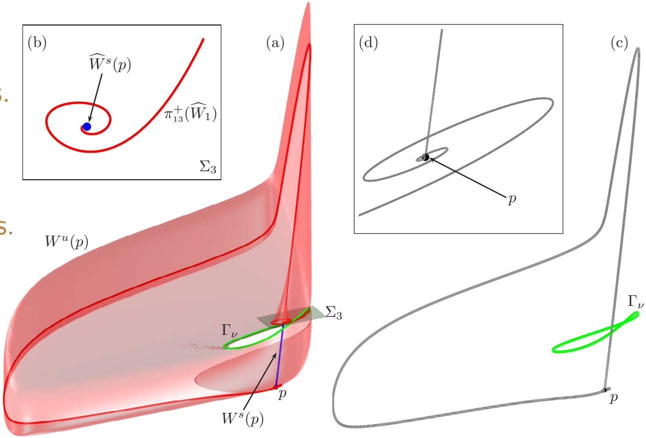
Interactions between $W^u(p)$ and S^r_ε

Consequence: Global return available + 1^7 MMO (linked) P.O.



Ensuing dynamics

- Period doubled MMOs.
- Long MMOs transients.
- New MMO signature.
- Shilnikov Homoclinic scenario.



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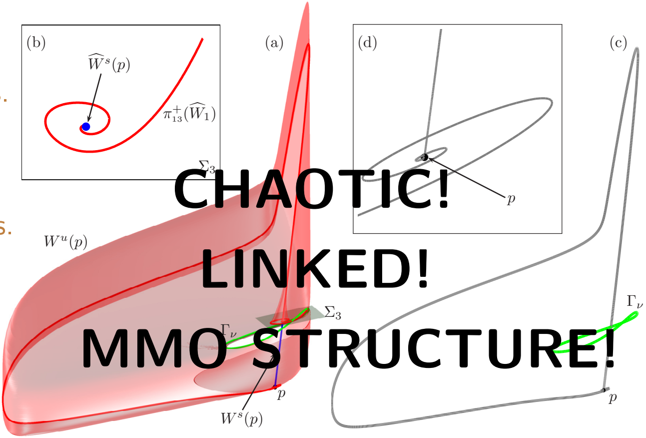


CHAOTIC!

LINKED!

MMO STRUCTURE!

Ensuing dynamics

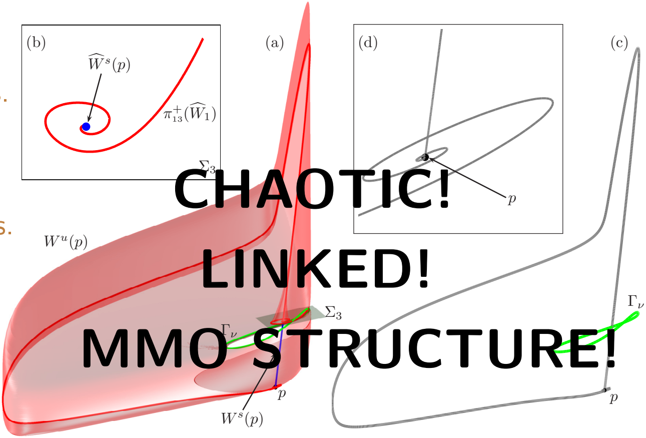
- Period doubled MMOs.

- Long MMOs transients.

- New MMO signature.

- Shilnikov Homoclinic scenario.

... and more.

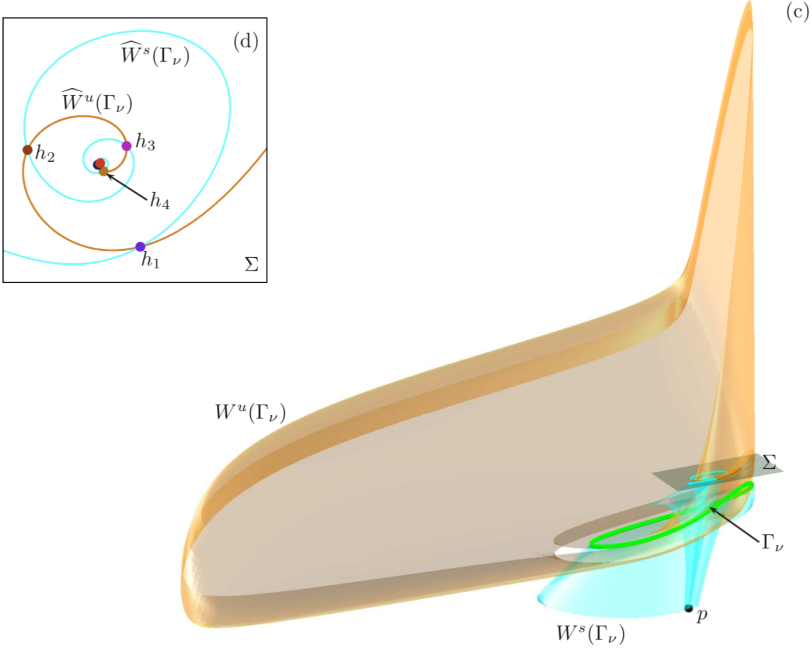


CHAOTIC!

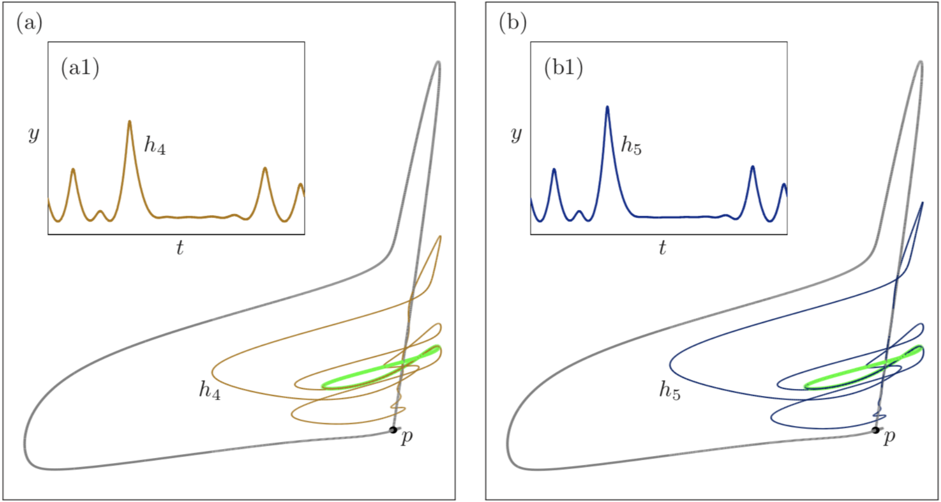
LINKED!

MMO STRUCTURE!

Homoclinic orbits to Γ_ν

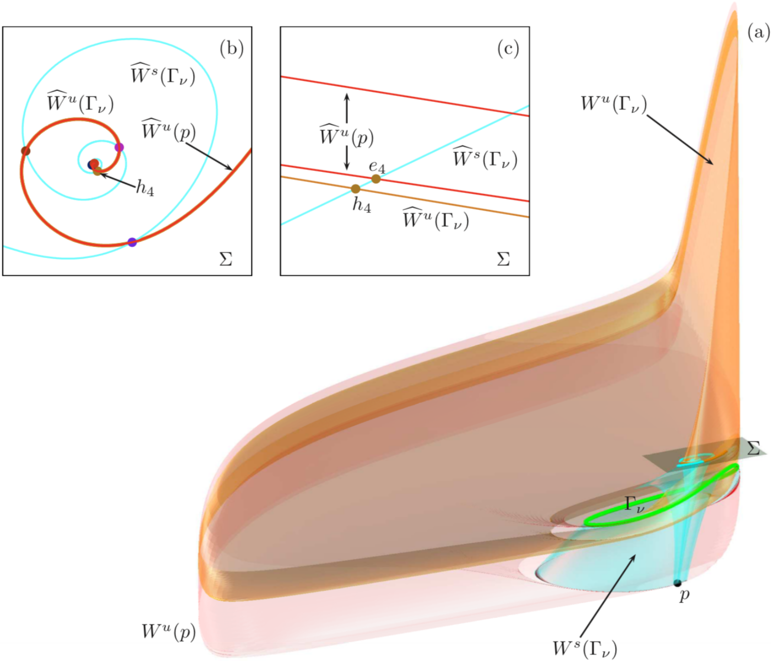


Homoclinic orbits to Γ_ν

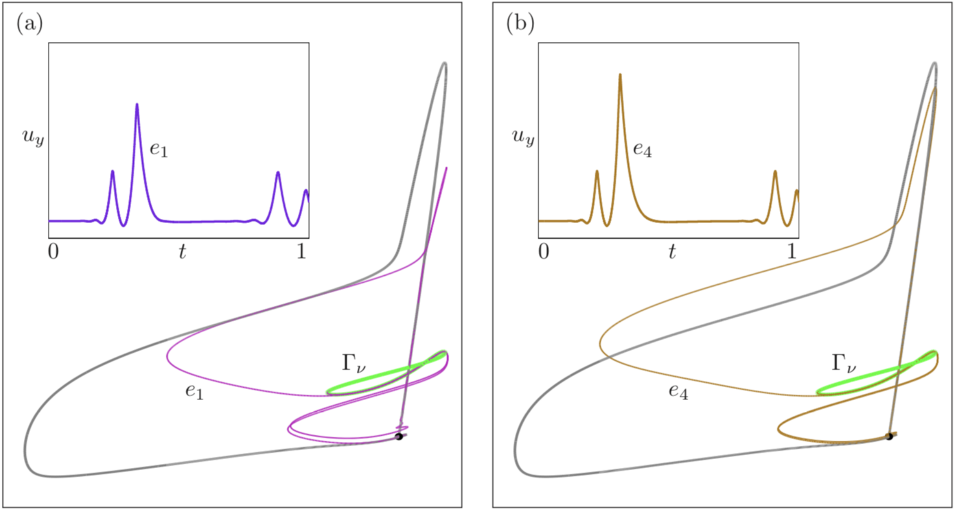


Close to EtoP!

Robust EtoP


(b)
$\widehat{W}^s(\Gamma_\nu)$
$\widehat{W}^u(\Gamma_\nu)$
$\widehat{W}^u(p)$
h_4
Σ
(c)
$\widehat{W}^u(p)$
e_4
$\widehat{W}^s(\Gamma_\nu)$
h_4
$\widehat{W}^u(\Gamma_\nu)$
Σ
(a)
$W^u(\Gamma_\nu)$
Σ
Γ_ν
p
$W^u(p)$
$W^s(\Gamma_\nu)$

Robust EtoP



To sum up...

- Interacting slow manifold and invariant manifold

- Period doubled MMOs.

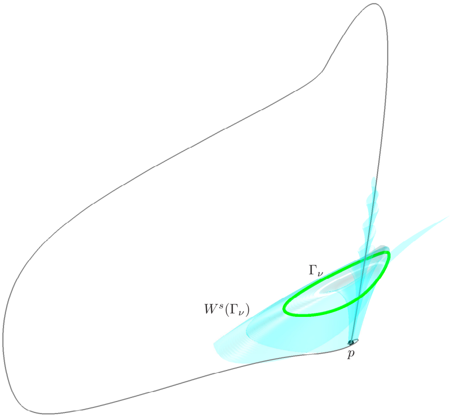
- Long MMOs transients.

- New observed MMO signature.

- Shilnikov homoclinic orbits.

- Homoclinic orbits to Γ_ν.

- Robust EtoP connections.



Future directions

- Codimension-one EtoP.

- Two-parameter bifurcation diagram.

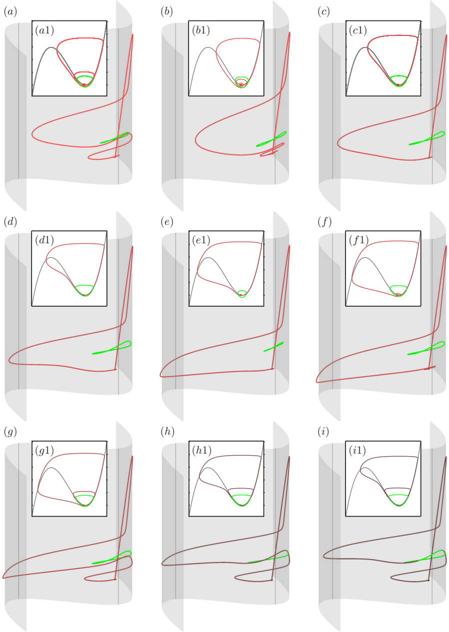
- Organizing center.

- Nonorientable manifolds and Shilnikov.

- Organization of Chaos.

- Chaotic attractor.

- Canard explosion of homoclinics.



[Ref: **J.M.**, B. Krauskopf, H. M. Osinga, SIAM J. Appl. Dyn. Syst., **17**(2) (2018).]