Oscillatory orbits in the planar three body problem

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The planar three body problem

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This talk is in the context of Celestial Mechanics:
According to wikipedia:
Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data.

More concretely we will talk about The planar three body problem
The planar three body problem describes the motion of 3 bodies q_1, q_2, q_3, of masses m_1, m_2, m_3 which move in a plane under the mutual gravitational forces.

The planar three body problem

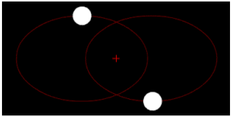
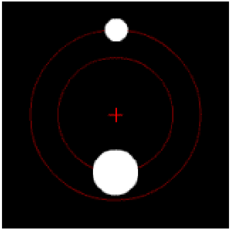
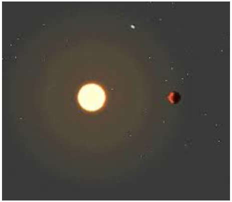
We want to understand the possible asymptotic motions (as $t \to \pm\infty$) of the bodies and answer classical questions like:

- Are the motion of three bodies stable (bounded)?
- Can it happen that some solutions become unbounded?
- Are there periodic motions? how many?
- Are there more surprising motions?
- Are there chaotic motions?

All these questions are related with what is known as:
the problem of stability.
In this talk we will not solve the three body problem and, therefore, we will not answer all the questions but we will provide some inside of some of them.
But first, let's talk about a simpler problem: The two body problem.

The two body problem

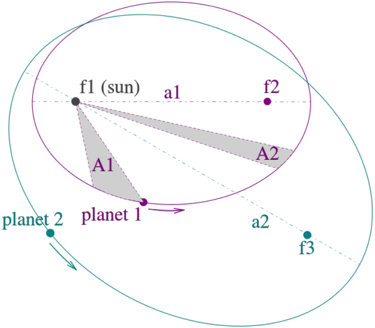
The two-body problem describes the motion of two bodies q_1, q_2, of masses m_1, m_2, moving in the space only under the influence of their mutual gravitational force. In classical mechanics, the two-body problem is to determine the motion of two point particles that interact only with each other. Common examples include a satellite orbiting a planet (ignoring other planets and the sun), a planet orbiting a star, two stars orbiting each other (a binary star), etc

,
,


The two body problem:Kepler laws (1609-1619)

In 1600 Johannes Kepler accepts the proposal to collaborate with the imperial astronomer Tycho Brahe. Their relation was a little "complicated" and, for this reason, Kepler had access to the complete data collected by Tycho only after his death. Those data were more complete that the ones from Copernicus.
To give a possible explanations to these data, Kepler elaborated his three laws, known now a days as Kepler laws.

The two body problem: Kepler laws (1609-1619)



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Kepler's laws describe the motion of a planet around the sun. It is a two body problem with q_1 being the sun. In the picture:

1. q_1 is the sun and q_2 is the planet 1
2. q_1 is the sun and q_2 is the planet 2

Kepler observed that the planet always satisfies some properties.

The two body problem: Kepler laws (1609-1619)

- The path of the planets about the sun is elliptical in shape, with the center of the sun being located at one focus. (The Law of Ellipses)
- An imaginary line drawn from the center of the sun to the center of the planet will sweep out equal areas in equal intervals of time. (The Law of Equal Areas)

  Consequently planets move faster when thay are closer to the sun.
- The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the sun. (The Law of Harmonies)

  The total orbit times for planet 1 and planet 2 have a ratio $\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$.

Newton's law of universal gravitation (1687)

• Any two bodies in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

• Is a general physical law derived from empirical observations by what Isaac Newton called induction.

Newton's law of universal gravitation (1687)

• Was formulated in Newton's work Philosophiae Naturalis Principia Mathematica ("the Principia"), first published on 5 July 1687.

• In modern language, the law states: Every point mass attracts every single other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between them.

• The first test of Newton's theory of gravitation between masses in the laboratory was the Cavendish experiment conducted by the British scientist Henry Cavendish in 1798.

It took place 111 years after the publication of Newton's Principia and 71 years after his death.

The two body problem: the equations

Second Newton's law: the vector sum of the external forces F on an object is equal to the mass m of that object multiplied by the acceleration vector a of the object: $F = ma$.

Applying Newton's law to the bodies, calling:

$q_1 = q_1(t) = (q_1^1(t), q_1^2(t), q_1^3(t)), \quad q_2 = q_2(t) = (q_2^1(t), q_2^2(t), q_2^3(t))$

its positions and recalling that the velocities are $v_i = v_i(t) = q_i'(t)$ and the accelerations $a_i = a_i(t) = q_i''(t)$, we obtain:

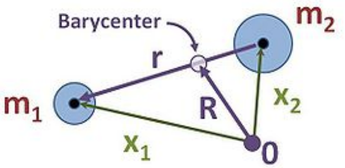
$$\begin{aligned} m_1 q_1''(t) &= G\frac{m_2 m_1 (q_2 - q_1)}{||q_2 - q_1||^3} \\ m_2 q_2''(t) &= G\frac{m_1 m_2 (q_1 - q_2)}{||q_1 - q_2||^3} \end{aligned}$$

where G is the gravitational constant.

• 6 second order differential equations which become a system of 12 first order non-linear differential equations!

The two body problem can be completely solved

- Goal: find trajectories $q_1(t)$ and $q_2(t)$ for all times t, given the initial positions $q_1(0)$, $q_2(0)$ and velocities $v_1(0)$ and $v_2(0)$.
- trick!: Adding and subtracting these two equations decouples them into two problems that can be solved independently.
- Adding the equations results in an equation describing the center of mass $R(t) = \frac{q_1(t)m_1+q_2(t)m_2}{m_1+m_2}$ (barycenter) motion.
- Subtracting the equations results in an equation that describes how the vector $Q(t) = q_1(t) - q_2(t)$ between the bodies evolves.
- If we know $R(t)$ and $Q(t)$ we can easily obtain the trajectories $q_1(t)$ and $q_2(t)$.



The motion of the center of mass

Consider the position of the center of mass:

$$R(t) = \frac{q_1(t)m_1 + q_2(t)m_2}{m_1 + m_2}$$

An easy computation gives:

$$\ddot{R} = 0$$

Therefore the velocity $V(t) = \dot{R}(t) = \frac{v_1(t)m_1+v_2(t)m_2}{m_1+m_2}$ of the center of mass is constant, from which follows that the total momentum

$$P(t) = m_1 v_1(t) + m_2 v_2(t)$$

is also constant (conservation of momentum).

The position $R(t)$ of the center of mass can be determined at all times from the initial positions and velocities.

The relative motion between the bodies

Consider the relative position of the bodies:

$$Q(t) = q_2(t) - q_1(t),$$

one can see that:

$$\ddot{Q} = G(m_1 + m_2)\frac{Q}{||Q||^3}$$

This is just the central forced problem or the Kepler problem, where a body is fixed at the origin and the other body moves as $Q(t)$.

Now we have $Q(t) = (Q^1(t), Q^2(t), Q^3(t))$.

• 3 second order differential equations which become a system of 6 first order non-linear differential equations!

Integrating the relative motion between the bodies

Using that the angular momentum: $G(t) = Q(t) \times \dot{Q}(t)$ satisfies $\dot{G} = 0$,we obtain that G is constant along the solutions.

- If $G = 0$, $Q(t)$ moves in a line: the motion is co-linear
- If $G \neq 0$ the motion is in a plane. We can take coordinates in such a way that $G = (0, 0, c)$ and then $Q^3(t) = \dot{Q}^3(t) = 0$ and we have the same second order equation, only with two variables $(Q^1(t), Q^2(t))$.
- We have 2 second order differential equations which become a system of 4 first order differential equations!

For experts:

A Hamiltonian system with 2 degrees of freedom with a first integral is integrable!

Integrating the relative motion between the bodies

Finally one can integrate the system and obtain that the orbits are given, using polar coordinates

$$Q(t) = (Q^1(t), Q^2(t)) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t))$$

as:

$$\rho = \frac{c^2}{\mu(1 + e\cos(\theta - \omega))}$$

where e and ω are constants of integration, only depend on the initial positions and velocities.

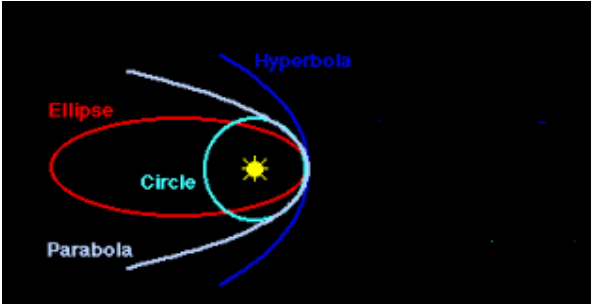
Possible motions:

- $e = 0$ motion is bounded: circles.
- $0 < e < 1$ motion is bounded: ellipses, where e is the excentricity.
- $e = 1$ motion is unbounded: parabolas
- $e > 1$ motion is unbounded: hyperbolas

Possible motions in the two body problem

We have seen that in the two body problem there are only three possible types of motion:

- Hyperbolic: $\|Q(t)\| \to \infty$ and $\|\dot{Q}(t)\| \to c > 0$ as $t \to \pm\infty$.
- Parabolic: $\|Q(t)\| \to \infty$ and $\|\dot{Q}(t)\| \to 0$ as $t \to \pm\infty$.
- Bounded (ellipses): $\limsup_{t\to\pm\infty} \|Q\| < +\infty$.



The planar three body problem

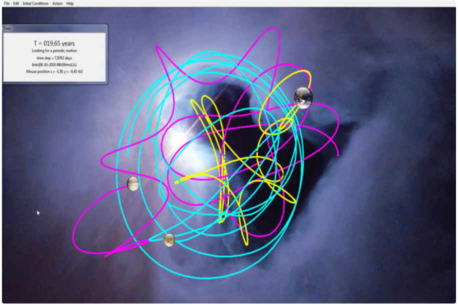
If we consider three bodies q_1, q_2, q_3, of masses m_1, m_2, m_3 Newton's laws give:

$$
\begin{aligned}
m_1\, q_1'' &= G\frac{m_1\, m_2(q_2-q_1)}{||q_2-q_1||^3} + G\frac{m_1\, m_3(q_3-q_1)}{||q_3-q_1||^3}\\
m_2\, q_2'' &= G\frac{m_2\, m_1(q_1-q_2)}{||q_1-q_2||^3} + G\frac{m_2\, m_3(q_3-q_2)}{||q_3-q_2||^3}\\
m_3\, q_3'' &= G\frac{m_3\, m_1(q_1-q_3)}{||q_1-q_3||^3} + G\frac{m_3\, m_2(q_2-q_3)}{||q_2-q_3||^3}
\end{aligned}
$$

- 9 second order differential equations which become a system of 18 first order differential equations!
- The conservation of the total angular momentum allows to consider the bodies moving on a plane: $q_i \in \mathbb{R}^2$.
- 6 second order differential equations which become a system of 12 first order differential equations!
- No enough first integrals to integrate and "predict" the motion of the three bodies!

The planar three body problem

- We want to understand the possible final motions in this case.
- We will see that in the case of three bodies the motion is richer!

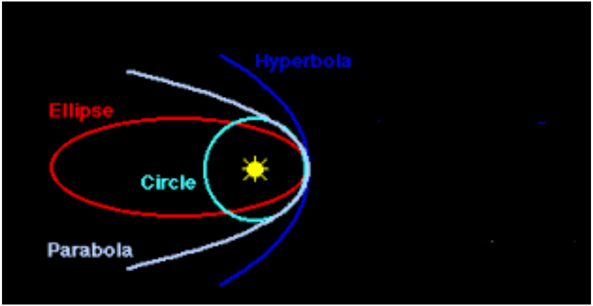


Asymptotic (final) motions

- Chazy (1922): Classification of all possible states that a 3BP can approach as $t \to \pm\infty$.
- Roughly speaking $q(t)$ refers to some relative position between the bodies:
  - $H^{\pm}$ (hyperbolic): $\|q(t)\| \to +\infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
  - $P^{\pm}$ (parabolic): $\|q(t)\| \to +\infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
  - $B^{\pm}$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
  - $OS^{\pm}$ (oscillatory):
    $\limsup_{t\to\pm\infty} \|q\| = +\infty$ and $\liminf_{t\to\pm\infty} \|q\| < +\infty$.
- Examples of all types except oscillatory already known by Chazy.

Asymptotic motions in the two body problem

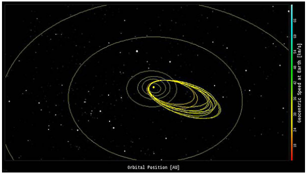
We have seen that in the case of two bodies here are only three types of motion: Hyperbolic, Parabolic, Bounded (ellipses)



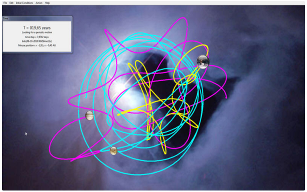
Oscillatory orbits can not exist if we only consider two bodies!

Asymptotic motions in the three body problem

We want to see that in the planar 3 body problem there exist initial conditions which give oscillatory motions.



These solutions travel close to the big elliptic orbits and are consequence of the existence of chaos in the system



The **Restricted** planar three body problem

$$m_1 q_1'' = G\frac{m_1 m_2(q_2-q_1)}{||q_2-q_1||^3} + G\frac{m_1 m_3(q_3-q_1)}{||q_3-q_1||^3}$$
$$m_2 q_2'' = G\frac{m_2 m_1(q_1-q_2)}{||q_1-q_2||^3} + G\frac{m_2 m_3(q_3-q_2)}{||q_3-q_2||^3}$$
$$m_3 q_3'' = G\frac{m_3 m_1(q_1-q_3)}{||q_1-q_3||^3} + G\frac{m_3 m_2(q_2-q_3)}{||q_2-q_3||^3}$$

Simplification: $m_3 \simeq 0$, put $m_3 = 0$ in the equations (after symplifying...):

$$\left.\begin{aligned} q_1'' &= G\frac{m_2(q_2-q_1)}{||q_2-q_1||^3} \\ q_2'' &= G\frac{m_1(q_1-q_2)}{||q_1-q_2||^3} \\ q_3'' &= G\frac{m_1(q_1-q_3)}{||q_1-q_3||^3} + G\frac{m_2(q_2-q_3)}{||q_2-q_3||^3} \end{aligned}\right\} \text{The motion of } q_1,\ q_2 \text{ is not affected by } q_3!$$

They form a two body problem

The restricted three body problem

• The two body problem is integrable: it satisfies Kepler laws.
q_1, q_2 can move in ellipses, hyperbolas or parabolas.
• We will put the solutions $q_1(t)$, $q_2(t)$ in the equations of q_3 and study the motion of q_3:

$$q_3'' = G\frac{m_1(q_1(t)-q_3)}{||q_1(t)-q_3||^3} + G\frac{m_2(q_2(t)-q_3)}{||q_2(t)-q_3||^3}$$

This is the restricted three body problem. q_1 and q_2 are called primaries.
• Assuming that the motion of q_3 occurs in the plane of rotation of the other two bodies, then the problem is known as
the restricted planar three-body problem (RP3BP)
• Is still not integrable!
• Our results are valid for the full three body problem, but in this talk I will focus in the restricted circular case.

The restricted planar three body problem (RP3BP)

Change of notation:

- We call the primaries q_S, q_J and the mass less body (comet) q
- We assume the two primaries $q_S(t)$, $q_J(t)$ move on ellipses (elliptic case): a particular case is when they move in circles (circular case).
  Typical models in the elliptic case with eccentricity e:
  - Sun–Jupiter–asteroid or comet: $e = 0.048$
  - Sun–Earth–Moon systems: $e = 0.016$

The equations of the RP3BP

- After normalizing, the motion of the comet $q = (q^1, q^2) \in \mathbb{R}^2$ is governed by 2 second order differential equations:

$$\frac{d^2 q}{dt^2} = \frac{(1-\mu)(q_S(t) - q)}{||q_S(t) - q||^3} + \frac{\mu(q_J(t) - q)}{||q_J(t) - q||^3},$$

where $\mu = \frac{m_2}{m_1 + m_2}$, is the mass ratio, and $0 \le \mu \le 1/2$, q_J and q_S are the known position of the primaries.

- Calling $p = (p^1, p^2) = \frac{dq}{dt}$ one obtain a system of 4 non-autonomous differential equations:

$$\begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= \frac{(1-\mu)(q_S(t) - q)}{||q_S(t) - q||^3} + \frac{\mu(q_J(t) - q)}{||q_J(t) - q||^3}, \end{aligned}$$

The equations of the RP3BP

This system can be written:

$$\begin{aligned}\frac{dq}{dt} &= \frac{\partial \mathcal{H}}{\partial p}(q, p, t; e, \mu) \\ \frac{dp}{dt} &= -\frac{\partial \mathcal{H}}{\partial q}(q, p, t; e, \mu)\end{aligned}$$

where

$$\mathcal{H}(q, p, t; e, \mu) = \frac{p^2}{2} - \frac{(1-\mu)}{||q - q_J(t)||} - \frac{\mu}{||q - q_S(t)||}.$$

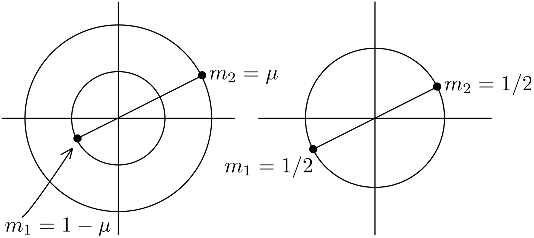
$p = (p^1, p^2)$ $q = (q^1, q^2)$.

This is a 2π-periodic in time Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian $\mathcal{H}$.

Parameters: $0 < e < 1$ the excentricity of the ellipse ($q_J(t)$ and $q_S(t)$ depend on e) and the mass ratio $\mu \in [0, 1/2]$.

The equations of the circular RP3BP

- To give the main ideas in this talk we will work in the circular case $e = 0$.
- In the circular case $e = 0$, the position of the primaries is simple and explicit:
  - The "big" body of mass $1 - \mu$ moves as: $q_S(t) = -\mu(\cos t, \sin t)$
  - The "small" body of mass μ moves as: $q_J(t) = (1 - \mu)(\cos t, \sin t)$.



The equations of the circular RP3BP in the circular case

Motion of the primaries in the circular case:

- The Hamiltonian is then:

$$\mathcal{H}(q,p,t;\mu) = \frac{\|p\|^2}{2} - \frac{1-\mu}{\|q+\mu(\cos t, \sin t)\|} - \frac{\mu}{\|q-(1-\mu)(\cos t, \sin t)\|}$$

  where $q, p \in \mathbb{R}^2$.
  It has a first integral, the Jacobi constant $\mathcal{J}(q,p,t;\mu)$.
- The Hamiltonian is 2π-periodic in time.
- Observation When $\mu = 1/2$, the two bodies move in the same circle at diametrally oposed points, therefore the Hamiltonian is π-periodic in time.
- To understand the behavior of the comet q, we have to deal with 4 non-linear, non-autonomous differential equations!

Simplification: RPC3BP in rotating coordinates

This non-autonomous Hamiltonian can be simplified:

- Fix the primaries at the x axis (periodic in time change of variables):

$$q_S = (\mu, 0), \quad q_J = (1-\mu, 0).$$

- We obtain an autonomous Hamiltonian:

$$H(q,p;\mu) = \frac{\|p\|^2}{2} - q \wedge p - \frac{1-\mu}{\|q+\mu(1,0)\|} - \frac{\mu}{\|q-(1-\mu)(1,0)\|}$$

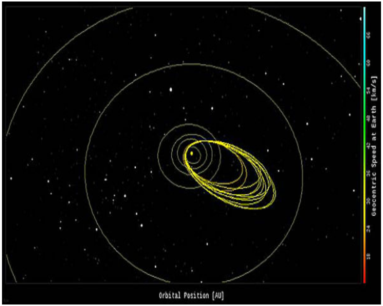
- Now the Hamiltonian is autonomous and therefore the energy H is a conserved quantity!
- (For experts): This corresponds to the conservation of the Jacobi constant

Therefore, fixing the energy level, the motion occurs in a three dimensional surface.

Oscillatory motions in the restricted planar circular three body problem (RPC3BP)

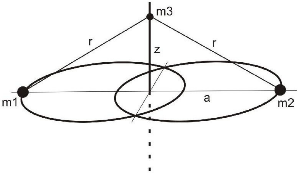
Goal: understand the motion of the massless body q under the influence of the other two q_S, q_J, which move, independently, in circles.

We will prove that, for some initial conditions, $q(t)$ has oscillatory motion.



Oscillatory motions in the Sitnikov problem

- Sitnikov (1960): restricted spatial elliptic three body problem.
- Existence of oscillatory motions when
  - Primaries have mass $\mu = 1/2$ and move on ellipses of small enough eccentricity.
  - Third body moves on the (invariant) vertical axis.



- Moser (1973) gave a new proof of Sitnikov results.

Oscillatory motions in the Non-restricted 3BP

- Alexeev (1970) extended the result to the non restricted spatial three body problem assuming the third mass small enough and using the simetries of the problem to reduce the dimension of the problem.
- Moeckel extended the result of Sitnikov to the case of three bodies with positive masses, two of them equal, in an isosceles configuration.
- Llibre and Simó found oscillatory motions for the collinear three body problem.

First results on Oscillatory motions for the RPC3BP

- First results by Llibre and Simó, 1980 following Moser's approach.

Theorem (Llibre-Simó)

Fix $\mu > 0$ small enough. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory.

- Their result requires that the mass ratio μ is extremely small
- J. Galante and V. Kaloshin (2011) prove the existence of orbits which initially are in the range of our Solar System and become oscillatory as $t \to +\infty$ with $\mu = 10^{-3}$ (realistic for the Jupiter-Sun).

Oscillatory motions for the planar 3 body problem

We present here some results in the restricted case and for the general case. Always for any value of the masses

- For the restricted planar circular case, for any values of the masses of the primaries.
- For the restricted planar elliptic case, for any values of the masses of the primaries and small enough eccentricity.
- For the general case, for any values of the masses of the bodies.
- In the restricted case the oscillatory motions we find have big angular momentum, therefore these orbits are away from collision.
- In the general case the total angular momentum is also big.

Results in the restricted circular case

Theorem (Guàrdia–Martín – S, Inv. Math. 2015))

Fix any $\mu \in (0, 1/2]$.
Then, there exists an orbit $(q(t), p(t))$ of the RPC3BP which is oscillatory.

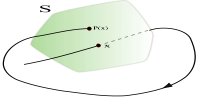
- In this work the eliminated the condition on μ proving that there exist oscillatory motions for every value of μ.
- The ideas contained in this work have been used to prove the more general result.

Theorem (Guàrdia–Martín – S, 2021))

There exists an orbit $q_1(t)$, $q_2(t)$, $q_3(t)$ of the planar three body problem which is oscillatory.

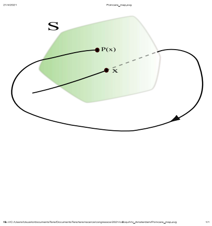
Oscillatory motions in the RCP3BP: Moser ideas

- We are talking about the solutions of a Hamiltonian system of 2 degrees of freedom, therefore the phase space is of dimension 4, variables (q_1, q_2) (position) and (p_1, p_2) (momenta).
- In a Hamiltonian system the Hamiltonian (energy) is preserved: $H(q(t), p(t)) = ct$, the motion occurs in dimension three: we can forget one variable, say p_2.
- Poincaré map P: we will look at the solutions every time they pass through section S.



We will look to the coordinate q_1 and the velocity p_1 every time we pass this section. This is the Poincaré map. It is a two dimensional map!

Oscillatory motions in the RCP3BP: Moser ideas



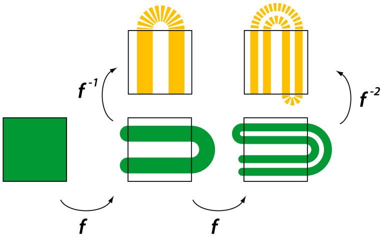
- We will study the dynamics of this map P on the section S to see that there are some orbits which are oscillatory.
- The oscillatory orbits will be a consequence of the existence of chaotic dynamics of this map P.

Oscillatory motions in the RCP3BP: Moser ideas

We want to prove that the Poincaré map P has chaotic behavior, that is, there exists an invariant set Σ such that $P_{|\Sigma}$ has chaotic dynamics:

- The set of periodic orbits is dense in Σ
- There is sensitive dependence of initial conditions
- There is a dense orbit in Σ

The more classical and simple map on $\mathbb{R}^2$ which presents chaotic behaviour is the horseshoe map (Smale, Fields medal 1966):

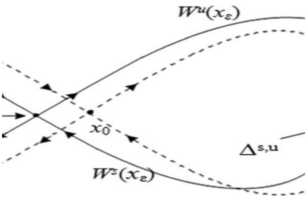


Oscillatory motions in the RCP3BP: Moser ideas

If we have a map P on the plane:

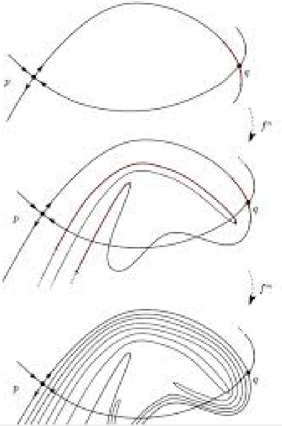
- A point z^* is called an fixed point of a map P if $P(z^*) = z^*$
- Hyperbolic fixed points ($DP(z^*)$ has no eigenvalues $|\lambda| = 1$) have stable and unstable manifolds (curves), which are invariant curves such that:

$$\begin{aligned} W^s(0,0) &= \{z \in \mathbb{R}^2, P^n(z) \to z^*, \text{ as } n \to \infty\}, \\ W^u(0,0) &= \{z \in \mathbb{R}^2, P^{-n}(z) \to z^*, \text{ as } n \to \infty\} \end{aligned}$$

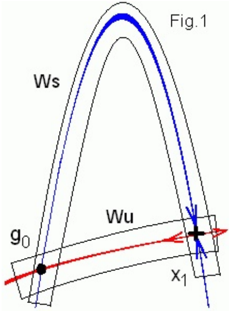


Oscillatory motions in the RCP3BP: Moser ideas

- Assume that the stable and unstable manifolds intersect transversally at some homoclinic point z_h.
- Then, using the classical Lambda-lemma, one can see the existence of a horseshoe for a suitable iteration of the Poincaré map:



Oscillatory motions: chaotic motions



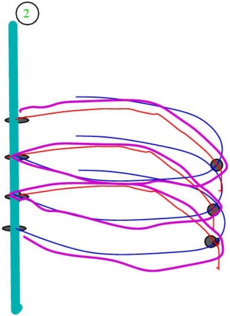
- Once we have the horseshoe map, one can apply classical results in Dynamical systems which provide the existence of symbolic dynamics.
- In particular we have periodic orbits af all the periods and also orbits which are dense in some invariant subset. These orbits approach the fixed point z^* and also the homoclinic intersection z_h!

Oscillatory motions in the RCP3BP: Moser ideas

- Recall that oscillatory orbits "approach" to infinity, therefore we need some good coordinates to study infinity.
- McGehee coordinates: $||q|| = \frac{1}{x^2}$, send infinity to zero. Infinity becomes the fixed point $z^* = (0,0)$ for the Poincaré map.
- $(0,0)$ is not hyperbolic; we have to see that it has stable and unstable curves $W^s(0,0)$, $W^u(0,0)$.
- We need to prove a Lambda-Lemma
- We also need to prove that these manifolds intersect at some point z_h.
- This will provide the existence of chaotic dynamics.
- We will have dense orbits
- These dense orbits are the oscillatory ones: they pass very close to the origin (the infinity) once and again but they go back near the point z_h.

Oscillatory motions: chaotic motions

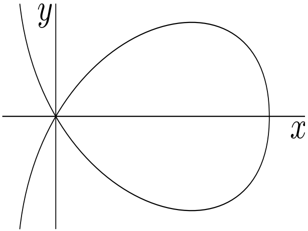
In the full dimensional space:



the oscillatory orbits pass very close to the origin (the infinity) once and again but they go back near the point z_h.

Oscillatory motions in the RCP3BP: Moser ideas

- Main difficulty in applying the approach to RPC3BP: prove the transversality of the invariant manifold of infinity for any value of the masses μ.
- For $\mu = 0$ (2- body problem between the comet and the sun) we have a fixed point $(0, 0)$ whose invariant manifolds coincide (parabolic orbits).



- What can we say when $\mu \neq 0$?

Transversality of the invariant manifolds of infinity in Llibre-Simó

- For $0 < \mu \ll 1$, expand in μ and compute the first order of the difference between the manifolds in terms of an integral (classical perturbative method known as Poincaré-Melnikov Theory) and an error of order $O(\mu^2)$.

$$d = \mu M(G_0) + O(\mu^2)$$

M is an explicit integral and depends on G_0, which is the angular momentum.

- We know only how to compute the Melnikov integral $M(G_0)$ provided $G_0 \gg 1$ and its size is $\mathcal{O}(e^{-G_0^3/3})$.

$$M(G_0) \simeq \mathcal{O}(e^{-G_0^3/3}).$$

- Llibre-Simó (Moser) method requires $\mu < e^{-G_0^3/3}$.
- using different methods we could eliminate the condition on μ proving that the invariant manifolds intersect transversally for every value of μ.
- The proof requires some sophisticated perturbative methods, and come ideas of complex analysis

The 3 body problem

In the planar 3BP, the three bodies $q_i \in \mathbb{R}^2$, $i = 1, 2, 3$ move influenced by each other. Hamiltonian formulation: $H(q, p) = K(p) - U(q)$, where

$$K(p) = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + \frac{1}{2m_3}p_3^2$$
$$U(q) = \frac{m_1 m_2}{\|q_1 - q_2\|} + \frac{m_1 m_3}{\|q_1 - q_3\|} + \frac{m_2 m_3}{\|q_2 - q_3\|}$$

Equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

The 3 body problem

- 6 d.o.f.
- Classical first integrals:
  - total linear momentum: $p_1 + p_2 + p_3$,
  - total angular momentum: $\det(q_1, p_1) + \det(q_2, p_2) + \det(q_3, p_3)$.
- After reduction, it becomes a 3 d.o.f.
- Fixing the energy in the reduced Hamiltonian reduces the dimension to 5 and then the Poincaré map in a suitable section becomes a 4-d map (instead, the Restricted Planar Circular 3BP is a 2-d map).
- Difficulties from the increased dimension:
  - Infinity is not a fixed point but a disc.
  - Prove the existence and regularity of the invariant manifolds of infinity
  - Compute the intersection of some invariant manifolds.
  - Obtaining 4-d isolating blocks to construct the horseshoe in higher dimension (a kind of high dimensional Lambda-lemma).