Chaos Theory: Determinism and Randomness in Nature and Mathematics

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Deterministic vs Random

flippping a coin
deterministic or random?

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Predictable vs Unpredictable

Predictable vs Unpredictable

Motion of the planets
-
Throwing an object (football, tennis...)

Weather & Climate
-
Turbulence
-
Coin Tossing

Deterministic vs Random ?

All these systems evolve according to PHYSICAL LAWS of CAUSE and EFFECT

Why is it that showers and even storms seem to come by chance, so that many people think it quite natural to pray for rain or fine weather, though they would consider it ridiculous to ask for an eclipse by prayer.

— Henri Poincaré

Science and Method (1908)

Henri Poincaré
1854–1912

Gottfried Leibniz
1646-1716

Isaac Newton
1643-1727

Developed "Differential Calculus", the foundation of the theory of "Differential Equations" and of "Dynamical Systems"

Mathematical models of physical systems

Gottfried Leibniz
1646-1716

Isaac Newton
1643-1727

The Mathematics of Differential Equations gave rise to the idea of DETERMINISM

The future evolution of system is completely determined by the physical laws and the initial condition of the system

Given [...] an intelligence which could comprehend all the forces by which nature is animated and the [...] positions of the beings which compose it, if moreover this intelligence were vast enough to submit these data to analysis, [...] nothing would be uncertain, and the future as the past would be present to its eyes.

— **Pierre-Simon Laplace**
Theorie Analytique de Probabilites (1812)

Pierre Laplace
1749-1827

> … it may happen that small differences in the initial conditions produce very great ones in the final phenomena.
>
> — **Henri Poincaré**
> Science and Method (1908)

Henri Poincaré
1854-1912

When flipping a coin, a small difference in how much force we apply can make it flip 8 times instead of 7 giving Heads instead of Tails

LORENZ EQUATIONS (1963): simplified model of the equations of weather.

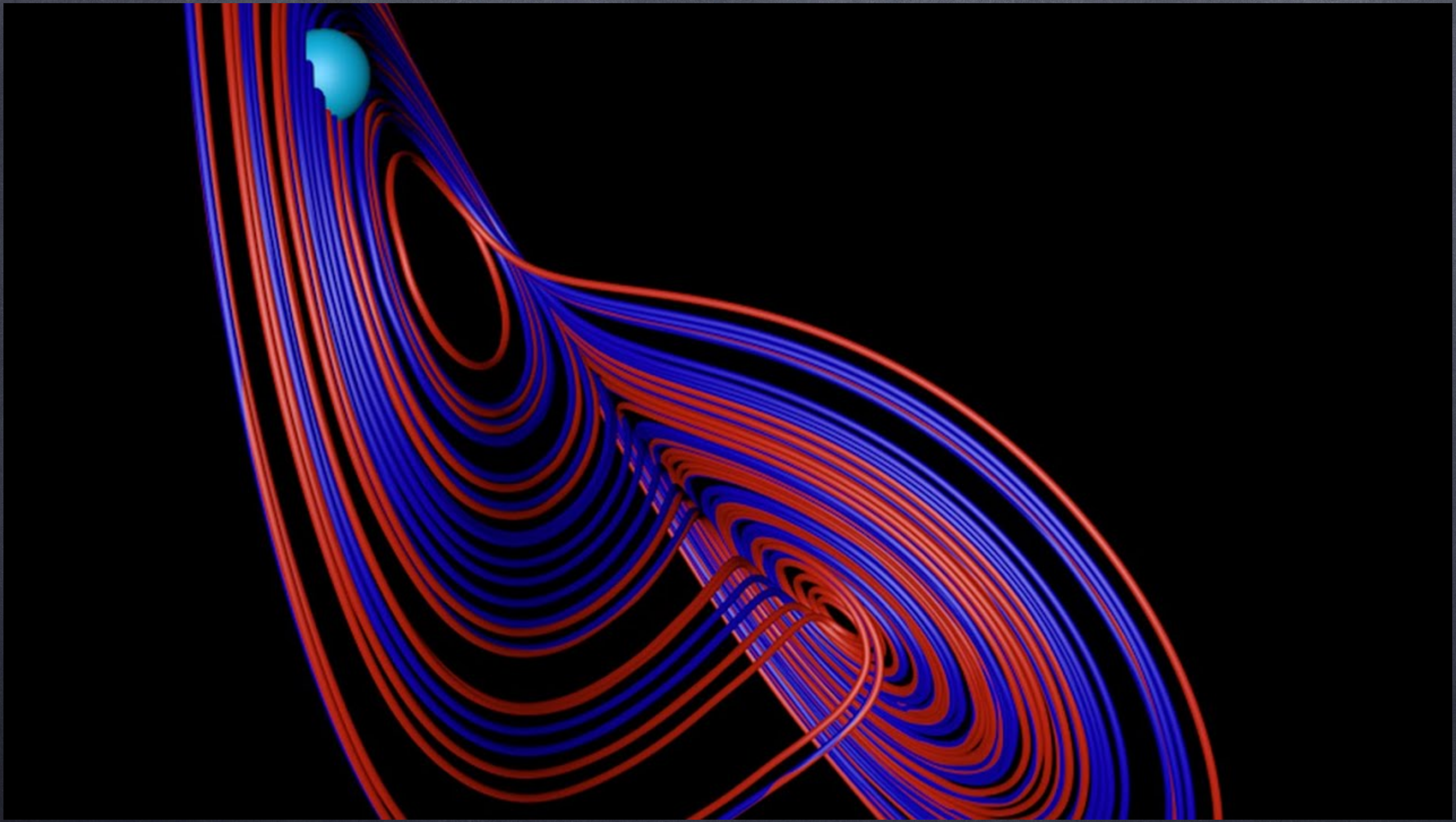
$$x' = 10(y - x)$$
$$y' = 28x - 28z - y$$
$$z' = xy - 8z/3$$

Edward Lorenz
Meteorologist
1917-2008

These equations are DETERMINISTIC:
the initial condition completely determines the future.

Lorenz observed a phenomena which he called SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

Computer simulation showing sensitive dependence on initial conditions.



Exactly what Poincaré had remarked

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

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1. Introduction

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

These modes of behavior may all be observed in the familiar rotating-basin experiments, described by Fultz, *et al.* (1959) and Hide (1958). In these experiments, a

Thus there are occasions when more than the statistics of irregular flow are of very real concern.

In this study we shall work with systems of deterministic equations which are idealizations of hydrodynamical systems. We shall be interested principally in nonperiodic solutions, i.e., solutions which never repeat their past history exactly, and where all approximate repetitions are of finite duration. Thus we shall be involved with the ultimate behavior of the solutions, as opposed to the transient behavior associated with

Lorenz did not have any supercomputers

Inputs and outputs were in the form of "punchcards" which produced sequences of numbers giving the coordinates of the trajectory

3.625 319	One day, he decided to restart the computations not from the last position computed the day before but from some previous position	3.625 000
-1.946		-1.946
-5.825		-5.825
....		....
9.425	To his dismay, the two sequences quickly diverged	4.725
-2.647		1.102

The computer's internal memory used 6 decimal places but only printed out 3. So the initial conditions were not exactly same

Simplest model

Consider the interval [0,1]

Suppose x in [0,1]

$f(x) = 10x \mod 1$

$x = 0.8362519...$

$f(x) = 0.362519....$

$f^2(x)=0.62519...$

$f^3(x)=0.2519..$

$f^4(x)=0.519...$

$f^5(x)=0.19...$

$f^6(x)=0.9...$

$f^7(x)=$???

$y = 0.8362521...$

$f(y) = 0.362521....$

$f^2(y) = 0.62521...$

$f^3(y) = 0.2521...$

$f^4(y) = 0.521...$

$f^5(y) = 0.21...$

$f^6(y) = 0.1...$

Even this very simple model satisfies sensitive dependence on initial conditions

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The unpredictability (CHAOS) is NOT randomness but sensitive dependence on initial conditions

Many chaotic systems satisfy a remarkable property:

Why are LONG TERM AVERAGES so PREDICTABLE?

Ergodic Theory is the study of Dynamical Systems from a STATISTICAL point of view

One of the main questions is: why do many chaotic systems have predictable long term averages

Theorem (Folklore, 1950's) f(x)=10x mod 1
Almost every trajectory has the SAME STATISTICAL distribution.

$$\lim_{n\to\infty} \frac{\#\{0 \le k < n : f^k(x) \in A\}}{n} = |A|$$

Lorenz Equations

$$x' = 10(y - x)$$
$$y' = 28x - 28z - y$$
$$z' = xy - 8z/3$$

CHAOTIC: sensitive dependence on initial conditions

Theorem (1960's–2000)
Almost every trajectory has the SAME STATISTICAL distribution.

Conclusions

When we say that a deterministic system, like flipping a coin, is "random" we are referring to two distinct characteristics:

- Short Term Unpredictability
- Long Term Statistical Predictability

Both of these can occur in deterministic systems

Determinism and Randomness are not opposites!

DETERMINISTIC refers to the MECHANISM

RANDOM refers to the OUTCOME

Thank You!