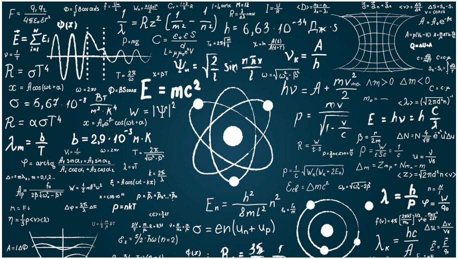
Perturbations of commutators

Martijn Caspers – TU Delft

at virtual Amsterdam

1. Motivation and origin of the problems



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$$[A, x] = Ax - xA,$$

of observables ($\approx$ self-adjoint matrices). Commutators tell how good two observables can be measured simultaneously (the smaller, the better).

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Heisenberg uncertainty principle (qualitative statement)

$[A, x] = 0$ if and only if A and x can be measured simultaneously.

Consequence: place Q and impulse P can never be determined with full accuracy at the same time $[P, Q] = i\hbar$.

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Can we control/estimate the perturbed commutator?

$$[A + \text{noise}, x] \preceq [A, x],$$

Whatever this means...

Functional calculus

Let A be a self-adjoint matrix.

For $p(x) = \sum_{k=0}^{n} \alpha_k x^k$ a polynomial we set

$$p(A) := \sum_{k=0}^{n} \alpha_k A^k.$$

For $f \in \mathbb{R} \to \mathbb{C}$ continuous we define

$$f(A) = \lim_i p_i(A)$$

where p_i are polynomials converging to f uniformly on compact sets.

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2. Kreĭn's problem: perturbations of commutators

M.G. Krein

Precise mathematical statement, at least going back to M.G. Kreĭn ($\approx$ 1964).

Is the following true?

Let $f : \mathbb{R} \to \mathbb{C}$ be Lipschitz. Is it true that for every self-adjoint A and x in $M_n(\mathbb{C})$:

$$\|[f(A), x]\| \leq C_{abs} \|f'\|_\infty \|[A, x]\|.$$

Here $C_{abs} > 0$ is some absolute constant.

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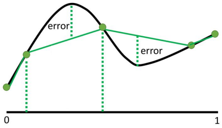
- Norm is the operator norm

$$\|y\| = \sum_{0 \neq \xi \in \mathbb{C}^n} \frac{\|y\xi\|}{\|\xi\|}.$$

  But also other norms shall be considered!
- The problem is hard if one asks for a constant C_{abs} independent of n.



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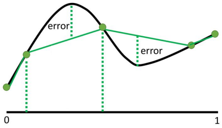
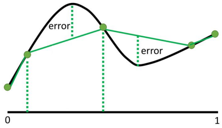


Lipschitz condition is essential

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$[A, x] = \begin{pmatrix} 0 & a-b \\ b-a & 0 \end{pmatrix}, \qquad [f(A), x] = \begin{pmatrix} 0 & f(a)-f(b) \\ f(b)-f(a) & 0 \end{pmatrix}.$$



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Then

$$[A, x] = \begin{pmatrix} 0 & a - b \\ b - a & 0 \end{pmatrix}, \qquad [f(A), x] = \begin{pmatrix} 0 & f(a) - f(b) \\ f(b) - f(a) & 0 \end{pmatrix}.$$

So if

$$\|[f(A), x]\| \leq C\|[A, x]\|$$

then

$$|f(a) - f(b)| \leq C|a - b|,$$

and so f is Lipschitz.

Is the following true?

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Take-home message of this talk:

Kreĭn's question can be resolved using harmonic analysis! Ingredients:

- Fourier multipliers and Calderón-Zygmund theory
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Definition: Let $\mathcal{S}_p$ be the Schatten-von Neumann L_p-space of $M_n(\mathbb{C})$.

It is $M_n(\mathbb{C})$ with norm

$$\|x\|_p := \mathrm{Tr}(|x|^p)^{1/p} := \mathrm{Tr}((x^* x)^{p/2})^{1/p}.$$

Sidenote: In general (for $B(H)$) $\mathcal{S}_p$ consists of all compact operators with singular value sequence in ℓ_p. The norm is the ℓ_p norm of these singular values.

Long non-exhaustive list of results on Kreĭn's problem:

$$\|[f(A), x]\|_p \leq C_p \|f'\|_\infty \|[A, x]\|_p, \qquad \forall A, x \in M_n(\mathbb{C}) \text{ self-adjoint, } f \text{ Lipschitz.}$$

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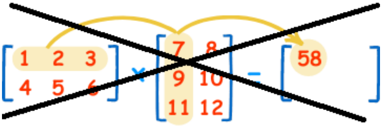
- Potapov, Sukochev 2011. True for any f Lipschitz (complete resolution).
- CMPS 2014, CPSZ 2019, CJSZ 2020. True for any f Lipschitz. Moreover,

$$C_p = C_{abs} p p^* = C_{abs} \frac{p^2}{p-1}.$$

3. Solving Kreĭn's question: Schur multiplication = entry-wise matrix multiplication



How first year students multiply matrices: dumb method.

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1st year student


2nd year student


Intrigued researcher

Where does Schur multiplication occur?

Consider the commutator

$$[A, x] = Ax - xA.$$

For A self-adjoint, we diagonalize (λ eigenvalues, p_λ eigenspace projections),

$$A = \sum_{\lambda \in \sigma(A)} \lambda\, p_\lambda.$$

We find the Schur (entry-wise) multiplication with matrix $(\lambda - \mu)_{\lambda,\mu}$ since

$$x = \sum_{\lambda,\mu \in \sigma(A)} p_\mu x p_\lambda.$$

$$[A, x] = \sum_{\mu \in \sigma(A)} \mu p_\mu x - \sum_{\lambda \in \sigma(A)} \lambda x p_\lambda = \sum_{\lambda,\mu \in \sigma(A)} (\mu - \lambda) p_\mu x p_\lambda.$$

The real problem!

The real problem! Recall $A = \sum_\lambda \lambda p_\lambda$. We have,

$$[f(A), x] = \sum_{\lambda,\mu\in\sigma(A)} (f(\mu) - f(\lambda)) p_\mu x p_\lambda = \sum_{\lambda,\mu\in\sigma(A)} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} (\mu - \lambda) p_\mu x p_\lambda$$

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Hence, solving Kreĭn's question boils down to showing that the Schur multiplier

$$T_{\phi_f} : y \mapsto \sum_{\lambda,\mu\in\sigma(A)} \phi_f(\mu, \lambda) p_\mu y p_\lambda$$

with symbol:

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is bounded on $\mathcal{S}_p := L_p(M_n(\mathbb{C}))$.

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Warning: estimating Schur multipliers can be extremely hard!

4. Solving Kreĭn's question: The transference method

Notation

- Set the gradient, now on the torus,

$$\nabla_{\mathbb{T}} = -i\frac{\partial}{\partial\theta}.$$

Set the trigonometric function $e_s(\theta) = e^{is\theta}$, $s \in \mathbb{Z}$.

For $\phi \in \ell_\infty(\mathbb{Z})$ we define the Fourier multiplier of $L_2(\mathbb{T})$,

$$\phi(\nabla_{\mathbb{T}})e_s = \phi(s)e_s, \qquad s \in \mathbb{Z}.$$

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- Set the gradient

$$\nabla = -i\frac{\partial}{\partial x}.$$

Set the trigonometric function $e_s(x) = e^{isx}$.

For $\phi \in C_b(\mathbb{R})$ we define the Fourier multiplier of $L_2(\mathbb{R})$,

$$\phi(\nabla)e_s = \phi(s)e_s, \qquad s \in \mathbb{R}.$$

Theorem (consequence of Caldéron-Zygmund theory)

Let $\psi : \mathbb{R}^2 \to \mathbb{C}$ be smooth on $\mathbb{R}^2 \backslash \{0\}$ and homogeneous, meaning

$$\psi(\lambda s, \lambda t) = \lambda \psi(s, t), \qquad \forall \lambda > 0, s, t \in \mathbb{R}.$$

Then

$$\psi(\nabla^2_{\mathbb{R}}) : L_p(\mathbb{R}^2) \to L_p(\mathbb{R}^2)$$

is bounded on L_p for $1 < p < \infty$.

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Then

$$\psi(\nabla_{\mathbb{R}}^2) : L_p(\mathbb{R}^2) \to L_p(\mathbb{R}^2)$$

is bounded on L_p for $1 < p < \infty$.

Moreover and highly non-trivial:

$$\mathrm{id}_n \otimes \psi(\nabla_{\mathbb{R}}^2) : L_p(M_n) \otimes L_p(\mathbb{R}^2) \to L_p(M_n) \otimes L_p(\mathbb{R}^2)$$

is bounded uniformly in n [Parcet '09, Cadilhac '18 or Bourgain 1980's].

Karel de Leeuw (1965)

Let $\psi : \mathbb{R} \to \mathbb{C}$ be continuous. Then,

$$\|\psi|_{\mathbb{Z}}(\nabla_{\mathbb{T}}) : L_p(\mathbb{T}) \to L_p(\mathbb{T})\| \leq \|\psi(\nabla_{\mathbb{R}}) : L_p(\mathbb{R}) \to L_p(\mathbb{R})\|$$

Remark: De Leeuw proves the analogous result for any discrete subgroup of $\mathbb{R}^n$.

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Remark: De Leeuw proves the analogous result for any discrete subgroup of $\mathbb{R}^n$.

Theorem (CPPR 15): De Leeuw's theorem holds for any discrete amenable subgroup Γ of a l.c. group G.

The unfortunate life story of Karel de Leeuw...

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(source: murderpedia.org)

Recall that solving Kreĭn's question boils down to showing boundedness of

$$T_{\phi_f} : y \mapsto \sum_{\lambda,\mu\in\sigma(A)} \phi_f(\mu,\lambda) p_\mu y p_\lambda$$

with symbol (not of Toeplitz form!):

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- Set

$$\pi : M_n(\mathbb{C}) \to L_\infty(\mathbb{T}^2) \otimes M_n(\mathbb{C}) : x \mapsto \sum_{\lambda,\mu} e_{(\mu-\lambda, f(\mu)-f(\lambda))} \otimes p_\mu x p_\lambda.$$

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  and $\psi_0(\lambda,\mu) = \frac{\lambda}{\mu}$ for $|\lambda| \leq |\mu|$.

- We have the magic formula,

$$(\psi_0(\nabla_{\mathbb{T}^2}) \otimes \mathrm{id}) \circ \pi = \pi \circ T_{\phi_f}.$$

  So Krein's problem is a matter of estimating $\|\psi(\nabla_{\mathbb{T}^2})\|_{cb}$, which is true by the previous 2 slides.

5. Consequence: non-commutative Lipschitz functions and Taylor approximation

$$
\begin{aligned}
f(x) &= \ln\left(1 + (\cos x - 1)\right) \\
&= (\cos x - 1) - \tfrac{1}{2}(\cos x - 1)^2 + \tfrac{1}{3}(\cos x - 1)^3 + O\left((\cos x - 1)^4\right) \\
&= \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^8\right)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + O\left(x^6\right)\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2} + O\left(x^4\right)\right)^3 + O\left(x^8\right) \\
&= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^6}{24} + O\left(x^8\right) \\
&= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + O\left(x^8\right).
\end{aligned}
$$

Theorem: Non-commutative Lipschitz functions

There exists a constant C_{abs} such that for any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$,any self-adjoint operators A and B in $M_n(\mathbb{C})$ and any $1 < p < \infty$ we have

$$\|f(A) - f(B)\|_p \leq C_{abs} \frac{p^2}{p-1} \|f'\|_\infty \|A - B\|_p.$$

Moreover, this estimate is sharp.

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Moreover, this estimate is sharp.

Proof: Take

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the result is just the inequality:

$$\|[f(C), x]\|_p \leq C_{abs} \frac{p^2}{p-1} \|f'\| \|[C, x]\|_p.$$

□

Several open directions:

- De Leeuw theorem for higher rank Lie groups like $SL_n(\mathbb{R}), n \geq 3$?
- Taylor expansions for functional calculus $\Rightarrow$ higher order approximations?
- Multi-linear (harmonic) analysis.
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Some special thanks go to collaborators (in this talk):

M. Junge, S. Montgomery-Smith, J. Parcet, M. Perrin, D. Potapov, E. Ricard, M. de la Salle, F. Sukochev, D. Zanin.