Equiangular lines & eigenvalue multiplicities

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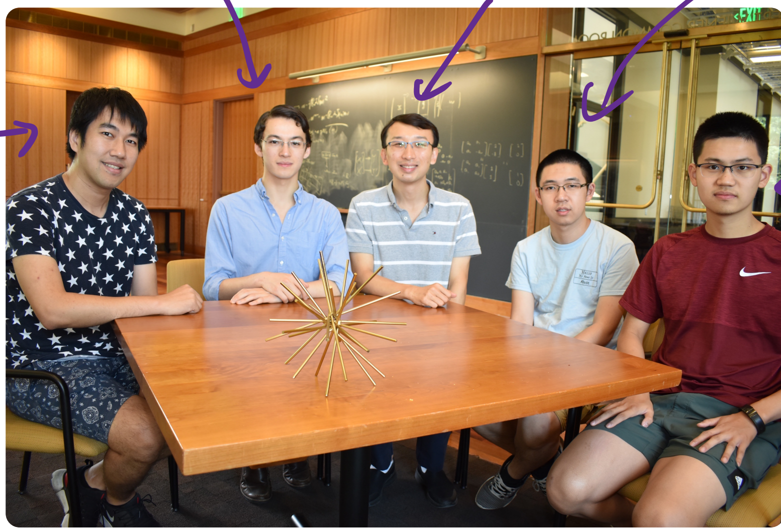

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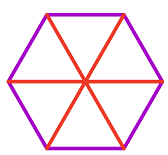
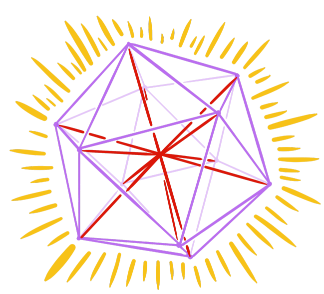
Equiangular lines

$N(d) = $ max # lines in $\mathbb{R}^d$ pairwise same angle

e.g. $N(2) = 3$

$N(3) = 6$





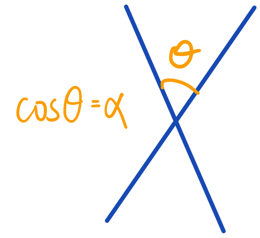
some constant $c > 0$

deCaen '00 $\quad c d^2 \le N(d) \le \binom{d+1}{2}$ $\quad$ Gerzon '73

in all constructions, pairwise angles $\to 90°$ as $d \to \infty$

Equiangular lines with a fixed angle

$N_\alpha(d) = \max \#$ equiangular lines in $\mathbb{R}^d$ with angle $\cos^{-1}\alpha$



Lemmens-Seidel '73 $\quad N_{1/3}(d) = 2(d-1) \quad \forall d \geq 15$

Neumann '73 $\quad N_\alpha(d) \leq 2d$ unless $\alpha = \frac{1}{\text{odd integer}}$

Neumaier '89 $\quad N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor \quad \forall d \geq d_0$

Next interesting case $\alpha = 1/7$?

Finally, we remark that the recent result of Shearer [13], that every number $t \geqslant t^* = (2+\sqrt{5})^{1/2} \approx 2.058$ is a limit point from above of the set of largest eigenvalues of graphs, makes it likely that the hypothesis of Theorem 2.6 can be satisfied if and only if $t < t^*$. (As communicated to me by Professor J. J. Seidel, Eindhoven, this has indeed been verified by A. J. Hoffman and J. Shearer.) Thus the next interesting case, $t = 3$, will require substantially stronger techniques.

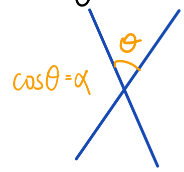
Some decades later ...

Bukh '16 $\quad N_\alpha(d) \leq C_\alpha d$

Balla-Dräxler-Keevash-Sudakov '18 $\quad N_\alpha(d) \leq 1.93d \quad \forall d \geq d_0(\alpha)$ if $\alpha \neq 1/3$

Problem: determine $\lim_{d\to\infty} \frac{N_\alpha(d)}{d}$

$N_\alpha(d)$ = max # equiangular lines with angle $\cos^{-1}\alpha$



Our work completely solves this problem

Lemmens-Seidel '73 $\quad N_{1/3}(d) = 2(d-1)$ $\quad \forall d$ suff. large

Neumaier '89 $\quad N_{1/5}(d) = \lfloor \frac{3}{2}(d-1) \rfloor$ $\quad \forall d$ suff. large

Our results $\quad N_{1/7}(d) = \lfloor \frac{4}{3}(d-1) \rfloor$ $\quad \forall d$ suff. large

$N_{1/9}(d) = \lfloor \frac{5}{4}(d-1) \rfloor$ $\quad \forall d$ suff. large

...

Thm (JTYZZ) $\quad \forall$ integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

And for other angles $\quad \forall$ fixed $\alpha \in (0,1)$

Set $\quad \lambda = \frac{1-\alpha}{2\alpha}$ (reparameterization) $\quad k = k(\lambda)$ "spectral radius order"

Then

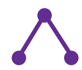
$$N_\alpha(d) = \begin{cases} \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor & \forall d \geq d_0(\alpha) \text{ if } k < \infty \\ d + o(d) & \text{if } k = \infty \end{cases}$$

$k(\lambda)$ = spectral radius order
$= \min k$ s.t. $\exists$ k-vertex graph G with $\lambda_1(G) = \lambda$

spectral radius of G
= top eigenvalue of adjacency mat of G

(set $k(\lambda) = \infty$ if $\nexists$ such G)

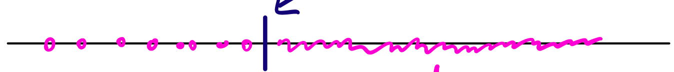
Examples

α	λ	k	G
$1/3$	1	2	
$1/5$	2	3	
$1/7$	3	4	
$1/(1+2\sqrt{2})$	$\sqrt{2}$	3	

$\lim_{d\to\infty} \frac{N_\alpha(d)}{d} = \frac{k(\lambda)}{k(\lambda)-1}$ was conjectured by Jiang – Polyanskii

who proved it for $\lambda < \sqrt{2+\sqrt{5}} \approx 2.058$

$\{\lambda_1(G) : G\} \subset \mathbb{R}$
adj mat



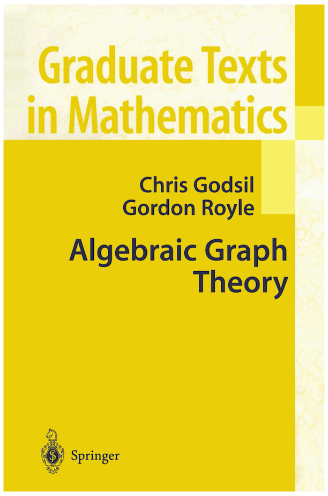
dense

Hoffman '72 + Shearer '89

New result on eigenvalue multiplicity

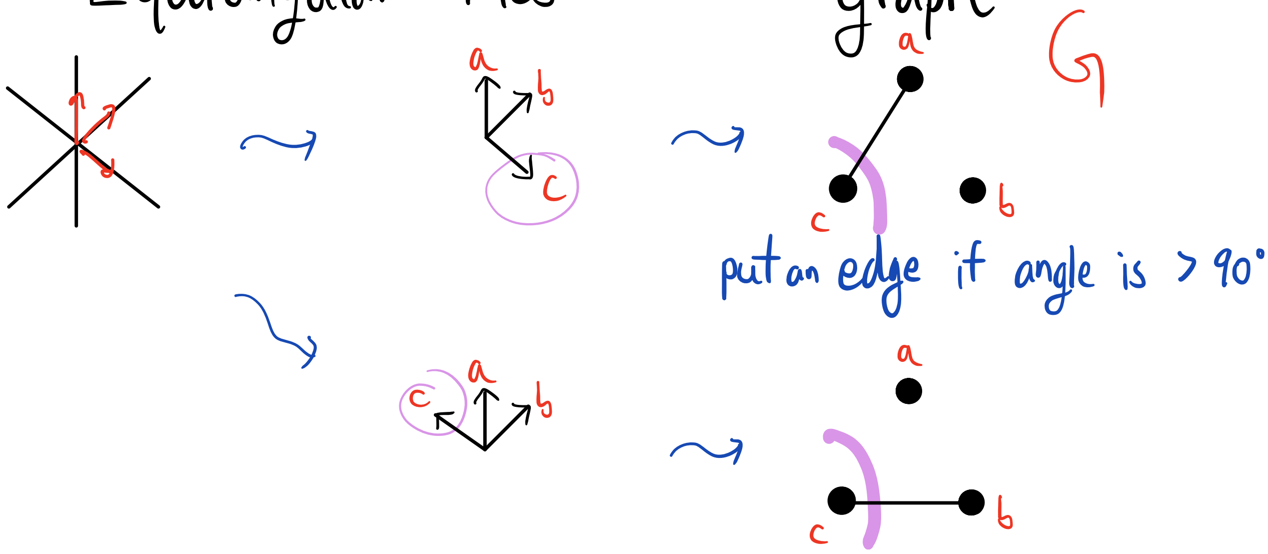
Thm [JTYZZ] Fix Δ. A connected n-vertex graph with max deg $\leq \Delta$ has second largest eigenvalue with multiplicity $O\left(\frac{n}{\log\log n}\right)$ ← sublinear $o(n)$

adj mat



The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In $\mathbb{R}^d$, for example, a simplex contains at most $d + 1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of $\mathbb{R}^d$, and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in $\mathbb{R}^d$ such that the angle between any two distinct lines is the same. We call this a set of *equiangular lines*. In this chapter we show how the problem of determining the maximum number of equiangular lines in $\mathbb{R}^d$ can be expressed in graph-theoretic terms.

Equiangular lines $\rightsquigarrow$ graph



Given a list of vectors, $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$

Gram matrix $\begin{pmatrix} \cancel{v_1 \cdot v_1}\ 1 & v_1 \cdot v_2\ \pm\alpha & \cdots \\ v_2 \cdot v_1\ \pm\alpha & \cancel{v_2 \cdot v_2}\ 1 & \cdots \\ \vdots & \vdots & \ddots\ 1 \end{pmatrix}$ rank $\leq d$

is a symmetric positive semidefinite matrix (all eigenvalues ≥ 0)

$= (1-\alpha)I - 2\alpha A_G + \alpha J$ since $v_i \cdot v_j = \pm\alpha$ for equiangular lines

I = $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, J = $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$

Convert to spectral graph theory problem:

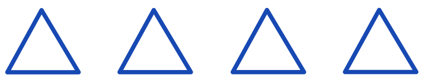
Given α, d, find graph G with max # vertices N s.t

$$(1-\alpha)I - 2\alpha A_G + \alpha J$$

is positive semidefinite & rank $\leq d$

Recall: $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for all suff. large d

Construction showing $N_{\frac{1}{5}}(9) \geq 12$

$G =$ 

$\alpha = \frac{1}{5}$

$$(1-\alpha)I - 2\alpha A_G + \alpha J = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\hline
\alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\hline
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha \\
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\hline
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & -\alpha & 1
\end{array}\right)$$

positive semidef & rank $= 9$

Upper bound on $N_\alpha(d)$

Rank-nullity theorem: $N = \text{rank} + \text{nullity}$

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & \\ & & & & \ddots \end{pmatrix}$$

$$\begin{aligned}
N &= \text{rank}\big((1-\alpha)I - 2\alpha A_G + \alpha J\big) + \text{null}\big((1-\alpha)I - 2\alpha A_G + \alpha J\big) \\
&\leq d + \text{null}\big((1-\alpha)I - 2\alpha A_G + \alpha J\big) \\
&\leq d + \underbrace{\text{null}\big((1-\alpha)I - 2\alpha A_G\big)}_{} + 1
\end{aligned}$$

$= \text{null}\left(\frac{1-\alpha}{2\alpha} I - A_G\right)$

$=$ multiplicity of $\frac{1-\alpha}{2\alpha}$ as an eigenvalue of G

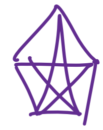
Since $(1-\alpha)I - 2\alpha A_G + \alpha J$ is pos semidef,

If $\frac{1-\alpha}{2\alpha}$ is an eigval of A_G, it must be either the

① largest eigval OR ② 2^{nd} largest

EASY HARD

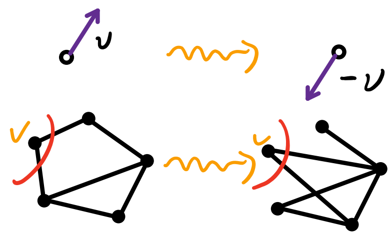
Q: Must all connected graphs have small 2^{nd} eigval multiplicity?



$4, -1, -1, -1, -1$

Not all graphs can arise from equiangular lines

Switching operation



Thm (Balla-Dräxler-Keevash-Sudakov)

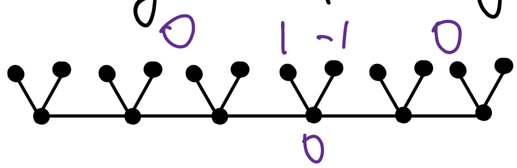
$\forall \alpha\ \exists \Delta$: can switch G to max deg $\leq \Delta$

Thm [JTYZZ] A connected n-vertex graph with max deg $\leq \Delta$ has second largest eigenvalue with multiplicity $O_\Delta\left(\frac{n}{\log\log n}\right)$ ⟵ sublinear $o(n)$

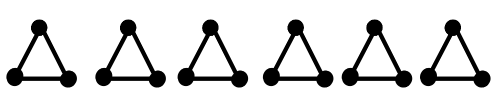
Near miss examples

- Strongly regular graphs
  e.g. complete graphs, Paley graphs

  Not bounded degree

- 

  mult$(0, G)$ linear, 0: a middle eigval



not connected

Open problem Max. possible 2^{nd} eigval multiplicity of a connected bounded degree graph?

Interesting to consider restrictions to (bdd deg)

- regular graphs
- Cayley graphs

For expander graphs, $\text{mult}(\lambda_2, G) = O\left(\frac{n}{\log n}\right)$

$N(A) \geq (1+c)|A| \quad \forall |A| \leq n/2$

For non-expanding Cayley graphs, $\text{mult}(\lambda_2, G) = O(1)$

Lee–Makarychev, building on Gromov, Colding–Minicozzi, Kleiner

Recently: McKenzie–Rasmussen–Srivastava

for a connected d-reg graph, $\text{mult}(\lambda_2, G) \leq O_d\left(\frac{n}{\log^{\frac{1}{5}-o(1)} n}\right)$

Lower bound constructions

- a Cayley graph on $PSL(2,p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{F}_p \;\; ad-bc=1 \right\} / \pm I$

(order $n \sim \frac{1}{2}p^3$)

gives 2^{nd} eigval multiplicity $\gtrsim n^{1/3}$

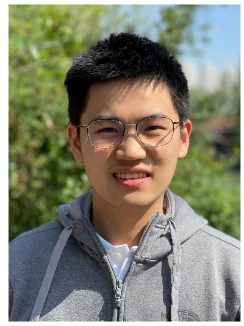
since all non-trivial representations have dim $\geq \frac{p-1}{2}$

(Frobenius)

- With

Milan Haiman

Carl Schildkraut



Shengtong Zhang

we constructed

- Connected bounded deg graph with 2^{nd} eigval mult

$$\gtrsim \sqrt{\frac{n}{\log n}}$$

- Conn. bdd deg Cayley graphs with 2^{nd} eigval mult

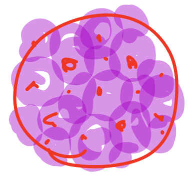
$$\gtrsim n^{2/5}$$

- group representations to get high mult.
- further manipulations to ensure 2^{nd} largest eigval

Open problem $\leq n^{1-c}$?

Thm If G is connected, n vtx, max deg $\leq \Delta$ then its 2nd largest eigval has multiplicity $O\left(\frac{n}{\log\log n}\right)$

Proof ideas



Lem (Net removal significantly reduces spectral radius)

If $H = G - (\text{an } r\text{-net of } G)$

then $\lambda_1(H)^{2r} \leq \lambda_1(G)^{2r} - 1$

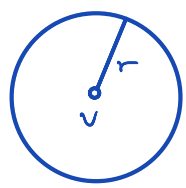
Pf $A_H^{2r} \leq A_G^{2r} - I$ entrywise ($A_H = A_G$ with the deleted edges zero'd)

to check diagonal entries, count closed walks

Suffice to exhibit a closed walk $v \circlearrowleft$ in G not in H

Lem (Local versus global spectra)

$$\sum_{i=1}^{|H|} \lambda_i(H)^{2r} \leq \sum_{v \in V(H)} \lambda_1\big(\underbrace{B_H(v,r)}_{r\text{-neighborhood}}\big)^{2r}$$



Pf

$\sum_{i} \lambda_i(H)^{2r}$ = # closed walks of length $2r$ in H

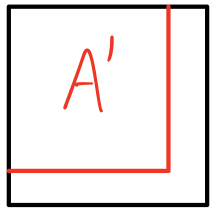
$= \sum_v$ #such walks starting at v (necessarily stays in $B_H(v,r)$)

$= 1_v^T A_{B_H(v,r)}^{2r} 1_v$

$\leq \lambda_1(B_H(v,r))^{2r}$

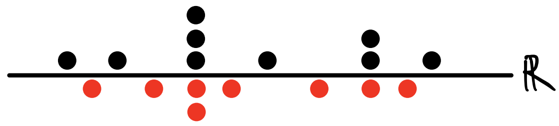
Tool: Cauchy eigenvalue interlacing theorem

Real sym matrix A



remove last row & column $\rightsquigarrow A'$

Then eigenvalues of A & A' interlace



$\Rightarrow$ Deleting a vertex cannot reduce $\text{mult}(\lambda, G)$ by more than 1

Proof sketch that $\text{mult}(\lambda_2, G) = o(n)$

assume all r-balls have spec rad $\leq \lambda := \lambda_2$

$H = G - (\text{a small } r_1\text{-net})$ $\qquad r_1 = c \log\log n$, $\quad r_2 = c \log n$

By local-global

$$\text{mult}(\lambda, H)\, \lambda^{2r_2} \leq \sum_i \lambda_i(H)^{2r_2} \leq \sum_{v \in V(H)} \underbrace{\lambda_1(B_H(v, r_2))}_{< \lambda - \varepsilon \text{ due to net-cutting}}{}^{2r_2}$$

$$\Rightarrow \quad \text{mult}(\lambda, H) = o(n)$$

By interlacing, $\text{mult}(\lambda, G) \leq \text{mult}(\lambda, H) + |\text{net}| = o(n)$

Summary:

- bound moment by counting closed $2r_2$-walks
- net removal significantly reduces local closed $2r_1$-walks
- relate these via local spectral radii

Limitations of trace method

"Approximate 2^{nd} eigval" multiplicity

Above proof shows $\leq O\left(\frac{n}{\log\log n}\right)$ eigenvalues within $O\left(\frac{1}{\log n}\right)$ of λ_2

[Haiman, Schildkraut, Zhang, Z.] A construction with a matching # of approx. 2^{nd} eigenvalues