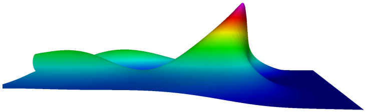
The mathematics behind nonlinear sound waves

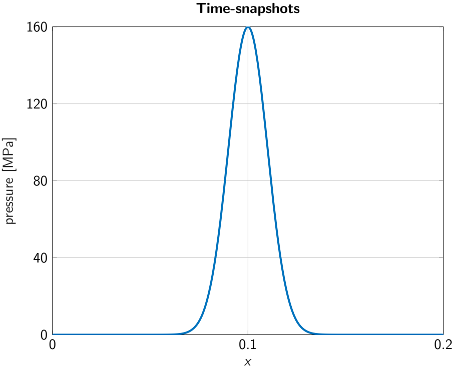
Vanja Nikolić

Radboud University



VU General Mathematics Colloquium, March 2021

Nonlinear sound propagation



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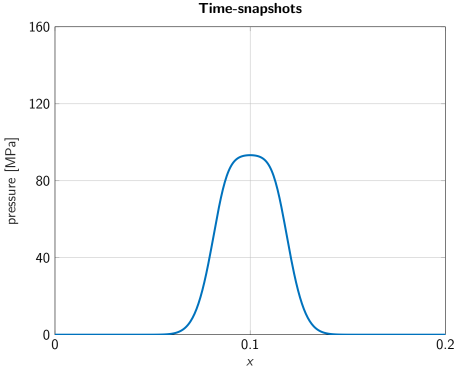

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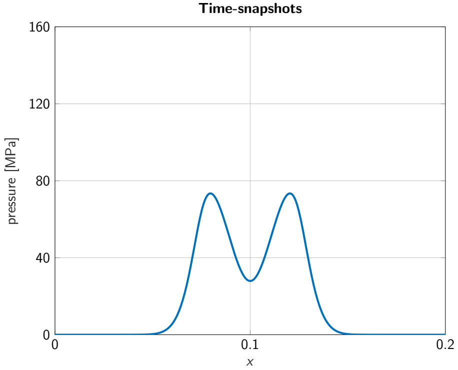
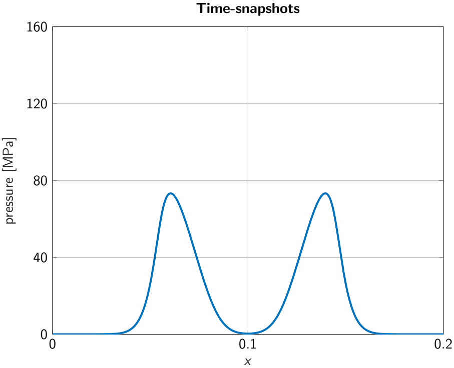


Nonlinear sound propagation



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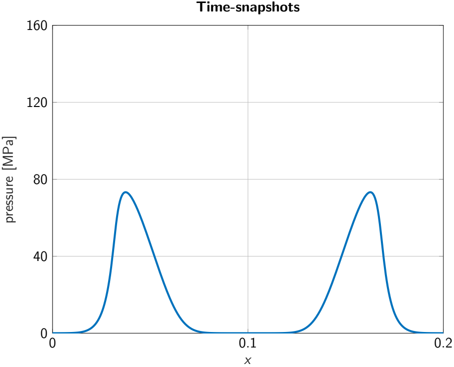

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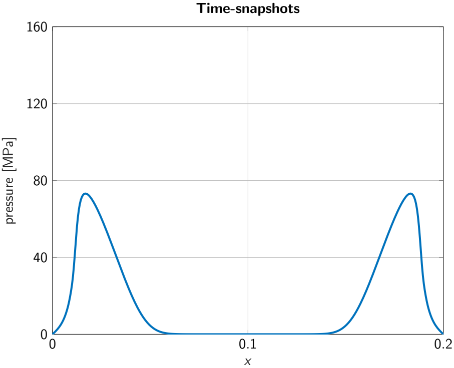
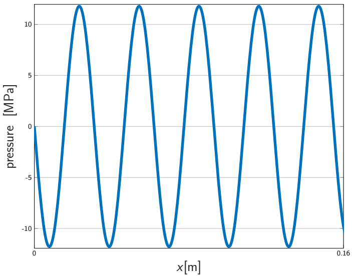
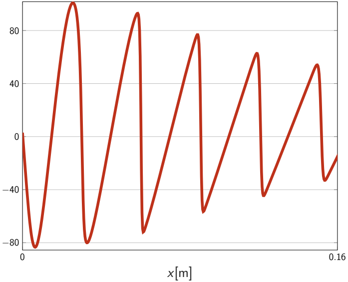


Linear vs. nonlinear waves

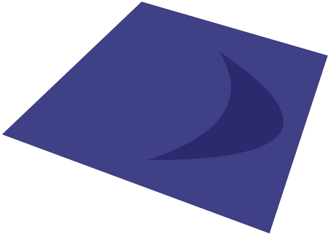




- Differences are pronounced with high amplitude-to-wavelength ratio

Possible applications

- High-Intensity Focused Ultrasound (**HIFU**)



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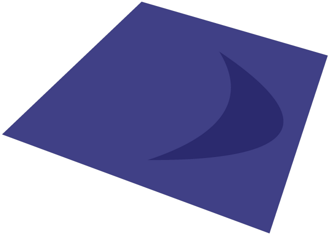
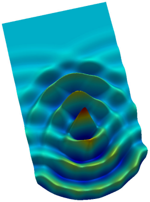
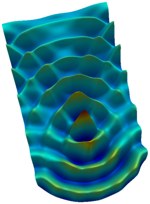
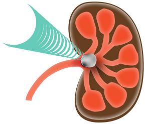


Possible applications

- Shock-wave lithotripsy







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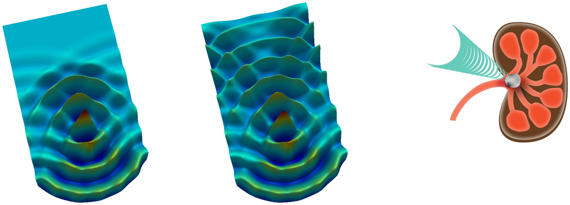

- Cancer treatments (under research)

A selection of references

- Modeling

  [Westervelt 1963], [Blackstock 1963], [Kuznetsov 1970], [Szabo 1993], [Jordan 2008], [Prieur, Holm 2011], [Christov, Christov, Jordan 2015], . . .

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- Mathematical analysis

  [Hughes, Kato, Marsden 1977], [Kawashima, Shibata 1992], [Mizohata, Ukai 1993], [Kaltenbacher, Lasiecka 2009, 2011, 2012], [Meyer, Wilke 2011], [Dörfler, Gerner, Schnaubelt 2015], . . .

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- Numerical analysis

  [Kaltenbacher, N., Thalhammer 2015], [N., Wohlmuth 2019], [Antonietti, Mazzieri, Muhr, N., Wohlmuth 2020], [Maier 2020], ...

Mathematical models in acoustics

Wave equation

$$u_{tt} - c^2 \Delta u = 0$$

Mathematical models in acoustics

Wave equation

$$u_{tt} - c^2\Delta u - \text{(damping)} = \text{(nonlinear effects)}$$

Mathematical models in acoustics

Westervelt's equation

$$u_{tt} - c^2 \Delta u - \quad b\Delta u_t \quad = \quad (ku^2)_{tt}$$

[Westervelt 1963]

Mathematical models in acoustics

↑ local nonlinear effects, relaxing/heterogeneous media, ...

Westervelt's equation

$$u_{tt} - c^2 \Delta u - \quad b\Delta u_t \quad = \quad (ku^2)_{tt}$$

↓ restricted propagation direction

[Westervelt 1963]

Mathematical models in acoustics

Westervelt's equation

$$u_{tt} - c^2\Delta u - \quad b\Delta u_t \quad = \quad (ku^2)_{tt}$$

$c > 0$ speed of sound, $b \geq 0$ sound diffusivity,

$k = \beta_a/(\varrho c^2)$, β_a coefficient of nonlinearity, ϱ mass density

[Westervelt 1963]

Physical background

Thermoviscous Navier–Stokes–Fourier system

1. Conservation of mass, momentum
2. Temperature law, Entropy production equation
3. Nonlinear pressure-density relation

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Thermoviscous Navier–Stokes–Fourier system

1. Conservation of mass, momentum
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- Weakly-nonlinear theory

  $\sim$ The governing system is approximated by one equation

More general models

- The Kuznetsov equation

$$\psi_{tt} - c^2 \Delta\psi - b\Delta\psi_t = \left(\kappa\psi_t^2 + |\nabla\psi|^2\right)_t$$

where $u = \varrho\psi_t$

- Incorporates local nonlinear effects

[Kuznetsov 1970]

More general models

- The Jordan–Moore–Gibson–Thompson equation

$$\tau\psi_{ttt}+\psi_{tt}-c^2\Delta\psi-(b+\tau c^2)\Delta\psi_t=\left(\kappa(\psi_t)^2+|\nabla\psi|^2\right)_t$$

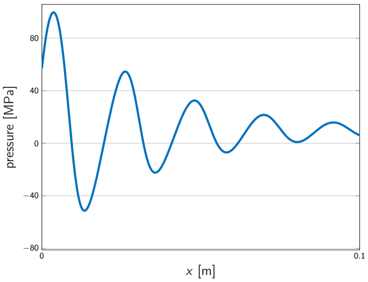
- Incorporates thermal relaxation

[Moore, Gibson 1960], [Thompson 1972], [Jordan 2008]

Analysis of nonlinear acoustic models

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (ku^2)_{tt}$$

- Influence of the damping $b = 1\,\mathrm{m}^2/\mathrm{s}$



Analysis of nonlinear acoustic models

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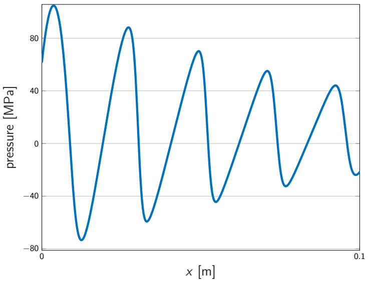

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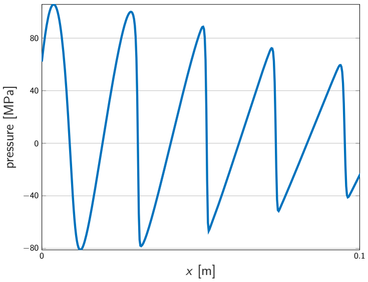


Limiting behavior

- Sound diffusivity b is in practice relatively small

- **Q:** Can we characterize the behavior of sound waves as $b \to 0$?

Challenges in the analysis

Westervelt's equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = (ku^2)_{tt}$$

Challenges in the analysis

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b\Delta u_t = 2ku_t^2 + 2kuu_{tt}$$

Challenges in the analysis

Westervelt's equation

$$(1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2ku_t^2$$

Challenges in the analysis

Westervelt's equation

$$(1 - 2ku)u_{tt} - c^2 \Delta u - b\Delta u_t = 2ku_t^2$$

- Quasi-linear wave equation
- Degenerates if $1 - 2ku = 0$

$\rightsquigarrow$ The pressure needs to be below $1/(2k)$

Limiting behavior

- Wave energy at time t

$$\|u\|_{\mathrm{E}}^2 = \frac{1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \frac{c^2}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x$$

Limiting behavior

- Wave energy at time t

$$\|u\|_{\mathrm{E}}^2 = \frac{1}{2}\int_\Omega |u_t|^2 \,\mathrm{d}x + \frac{c^2}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

- How fast does u converge in the energy norm as $b \to 0$?

Limiting behavior

- Convergence in the energy norm

$$\sup_{t\in(0,T)} \|u^{(b)}(t) - u^{(0)}(t)\|_{\mathrm{E}} \lesssim b$$

[Kaltenbacher & N. 2020]

- Assumptions: Smooth and small data, short final time T
- Smallness comes from ensuring that pressure stays below $1/2k$

Limiting behavior of more general models

- The Jordan–More–Gibson–Thompson equation

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (b + \tau c^2)\Delta\psi_t = f(\psi_t, \psi_{tt}, \nabla\psi, \nabla\psi_t)$$

- Perturbation of the wave speed for

$$z = \tau\psi_t + \psi$$

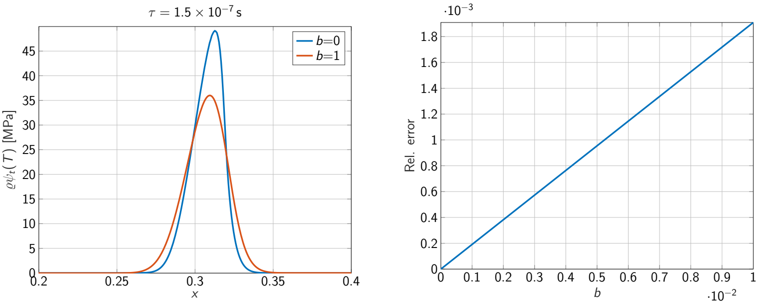
- Compared to Westervelt's equation: Weaker regularity assumptions

Limiting behavior of more general models

- Convergence in the energy norm for the JMGT equation

$$\sup_{t\in(0,T)} \|\psi^{(b)}(t) - \psi^{(0)}(t)\|_{\mathsf{E}} \lesssim b$$

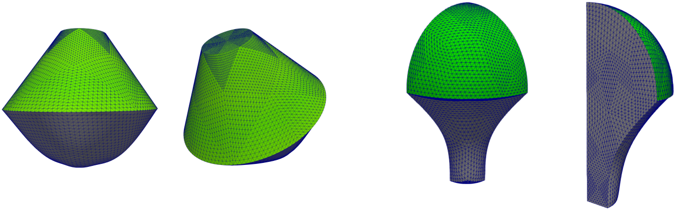
[Kaltenbacher & N. 2021]



Numerical approximation

Semi-discrete acoustic models

$$u_{h,tt} - c^2\Delta_h u_h - \quad (\text{damping})_h \quad = \quad (\text{nonlinear effects})_h$$



Examples of computational domains

Numerical analysis

1. Does an approximate solution u_h exist?

2. Is u_h a stable and accurate representation of u?

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3. How fast does u_h converge as $h \to 0$?

Challenges

- Approximate pressure is not very smooth in general

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- Approximate pressure is not very smooth in general

  $\rightsquigarrow$ The arguments from the continuous setting cannot be transferred

- The approximate pressure needs to be below $1/(2k)$ as well

Strategy: A fixed-point argument

- Linearized problem

$$(1 - 2k w_h) u_{h,tt} - c^2 \Delta_h u_h - b \Delta_h u_{h,t} = 2k w_{h,t} u_{h,t}$$

Define the mapping $\mathcal{F} : w_h \mapsto u_h$, where

- w_h in an h neighborhood of u
- u_h solves the linearized problem

The **fixed point** $w_h = u_h$ solves the nonlinear problem.

Numerical analysis

- A priori error estimate

$$\|u - u_h\|_E \lesssim h^p$$

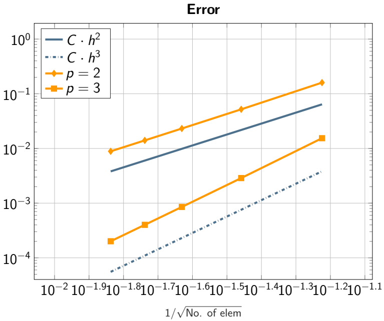
[N. & Wohlmuth 2019], [Antonietti, Mazzieri, Muhr, N., & Wohlmuth 2020]

- Assumptions: Small and smooth data and small enough h

Numerical analysis

- Westervelt's equation with a given source term

  ⇝ Exact solution known

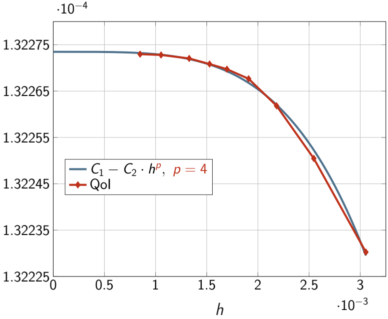


[Antonietti, Mazzieri, Muhr, N., & Wohlmuth 2020]

Numerical analysis

- Westervelt's equation with a sinusoidal boundary excitation

  $\rightsquigarrow$ Unknown solution



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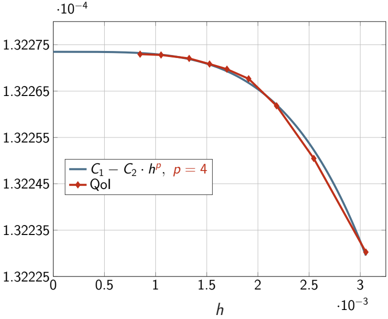


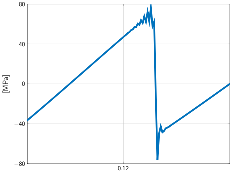
The degeneracy bound $1/(2k) \approx 214$ MPa

[Antonietti, Mazzieri, Muhr, N., & Wohlmuth 2020]

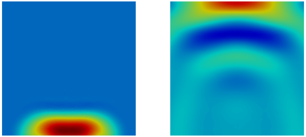
About numerical simulations

- Computationally expensive

- Non-physical oscillations around peaks



- Unwanted reflections



Reflective vs. absorbing boundaries

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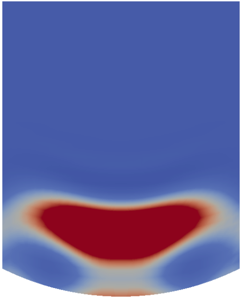

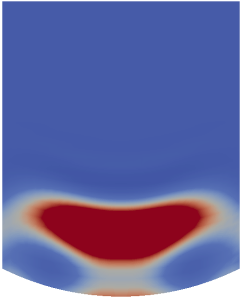

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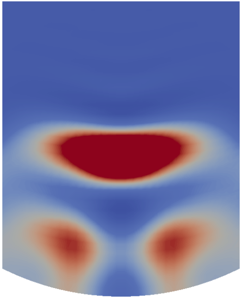 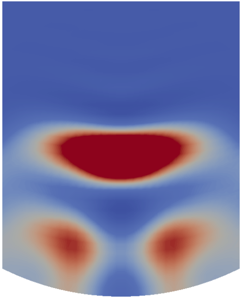

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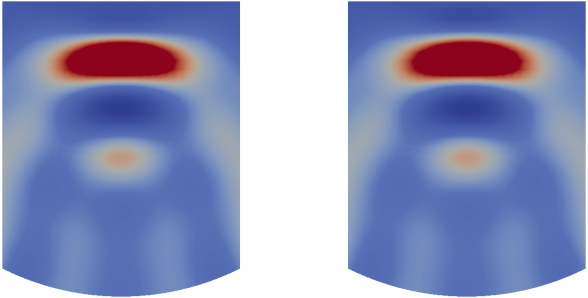

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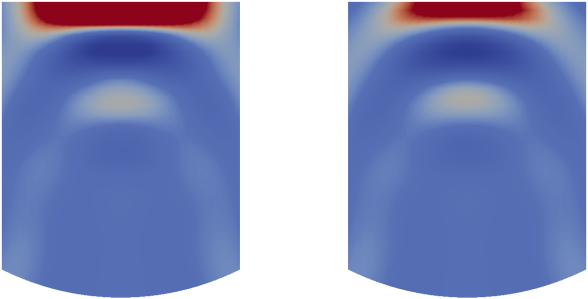

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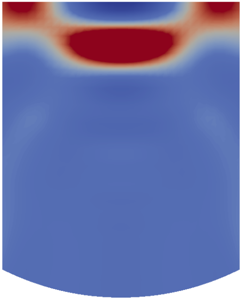

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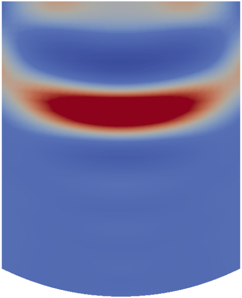
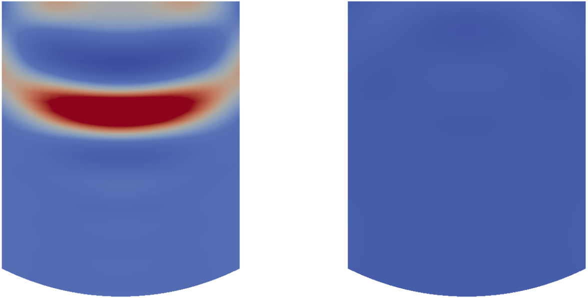


Unsuitable conditions $\rightsquigarrow$ Pollution of the pressure field

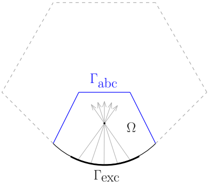
Reflective vs. absorbing boundaries

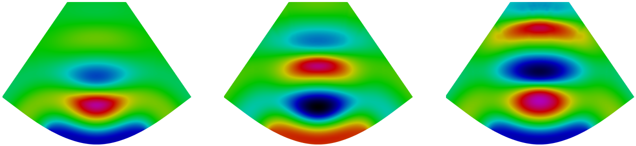


A solution: Problem-tailored absorbing conditions

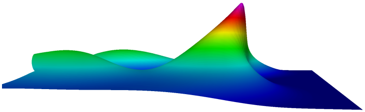
[Shevchenko & Kaltenbacher 2015], [Muhr, N., & Wohlmuth 2019]

Combined with adaptivity

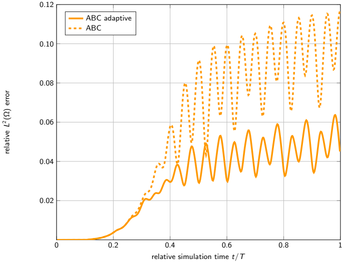




- Focusing



- Improvement in the relative error: 4% vs. 8%



[Muhr, N., & Wohlmuth 2019]

Summary

- PDE-based analysis & simulation offers a way of better understanding (nonlinear) ultrasonic waves

Further topics

- Coupling strategies (elasto-acoustic)
- Propagation in general lossy media
- Manipulation of sound waves

Based on

- V. Nikolić and B. Wohlmuth,
  A priori error estimates for the finite element approximation of Westervelt's quasi-linear acoustic wave equation,
  SIAM J. Numer. Anal., 2019.
- P. F. Antonietti, I. Mazzieri, M. Muhr, V. Nikolić and B. Wohlmuth,
  A high-order discontinuous Galerkin method for nonlinear sound waves,
  J. Comput. Phys., 2020.
- B. Kaltenbacher and V. Nikolić,
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