Delayed feedback stabilization of periodic orbits and spatio-temporal patterns

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Pyragas control [Pyragas '92]

system with unstable periodic orbit

$x_*(t+p) = x_*(t)$

NONINVASIVE control term

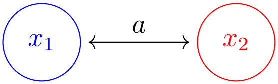
$$\dot{x}(t) = f(x(t)) + B\,[x(t) - x(t-p)]$$

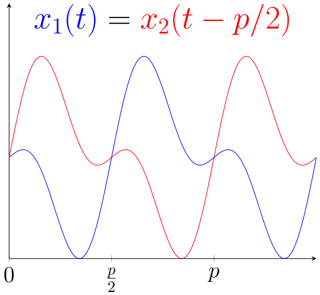
gain matrix or control gain

time delay

Is there a matrix B that makes x_* stable?

Spatio-temporal patterns: example





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$$\dot{x}_2(t) = f(x_2(t)) + a(x_2(t) - x_1(t))$$

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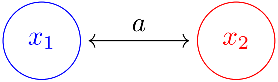
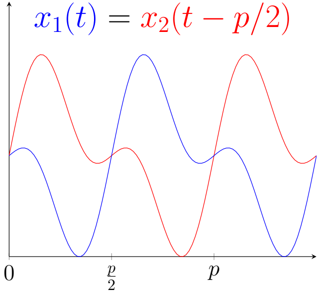




$$\dot{x}_1(t) = f(x_1(t)) + a(x_1(t) - x_2(t)) + B\left[x_1(t) - x_2\left(t - \frac{p}{2}\right)\right]$$
$$\dot{x}_2(t) = f(x_2(t)) + a(x_2(t) - x_1(t)) + B\left[x_2(t) - x_1\left(t - \frac{p}{2}\right)\right]$$

cf. [Nakajima, Ueda, '98] and [Fiedler, Flunkert, Hövel, Schöll, '10]

Method of steps

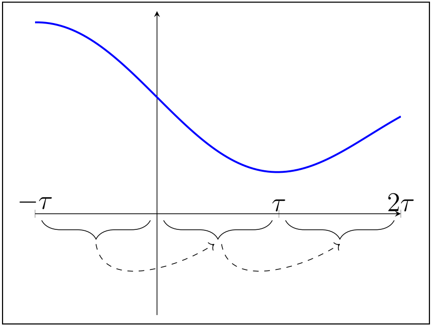
Consider

$$\dot{x}(t) = g(x(t), x(t-\tau))$$

with $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$,
one time delay $\tau > 0$ and
initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0]$$

for $\phi \in C\left([-\tau, 0], \mathbb{R}^N\right)$.



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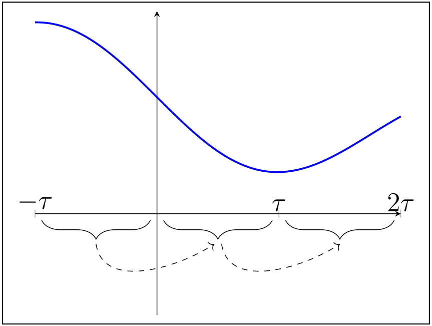

For $t \in [0, \tau]$:

$$\dot{x}(t) = g(x(t), \phi(t-\tau)).$$

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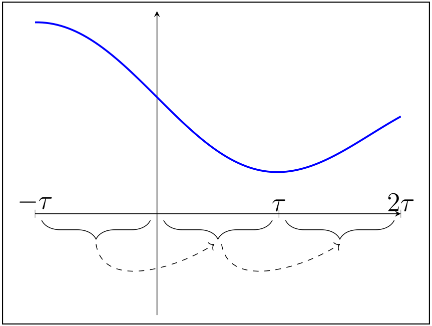

For $t \in [0, \tau]$:

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$\Longrightarrow$ solve and repeat.

Delay differential equations as dynamical systems

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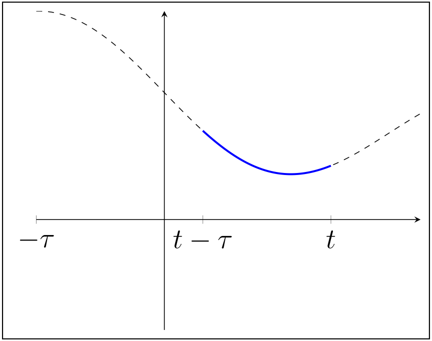


Generate semiflow $S_t(\phi)$ with $t \geq 0$ and $\phi \in C\left([-\tau, 0], \mathbb{R}^N\right)$ by

- computing solution forwards
- updating history segment $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$.

Delay differential equations as dynamical systems

Consider

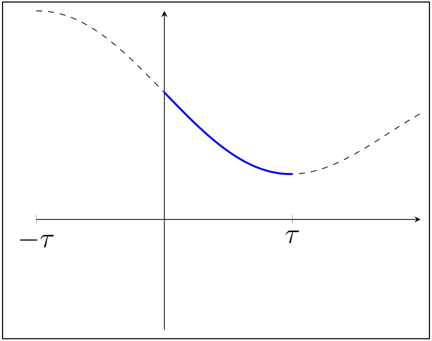
$$\dot{x}(t) = G(x_t)$$

with history segment

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0]$$

and right hand side

$$G : C\left([-\tau, 0], \mathbb{R}^N\right) \to \mathbb{R}^N.$$



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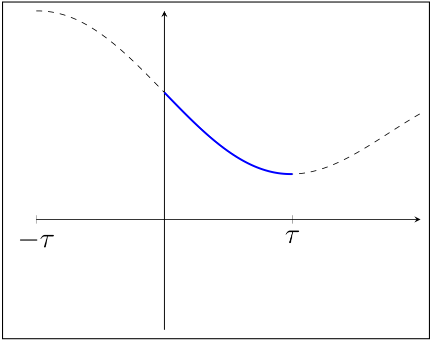


Semiflow $S_t(\phi) \in C\left([-\tau, 0], \mathbb{R}^N\right)$ with $t \geq 0$ and $\phi \in C\left([-\tau, 0], \mathbb{R}^N\right)$

$S_t(\phi)$ is a C^1 function if $t \geq \tau$.

Let x_* be a p-periodic solution of

$$\dot{x}(t) = f(x(t), t) + B\left[x(t) - x(t-p)\right]$$

with $f(\,.\,, t+p) = f(\,.\,, t)$. The linearized equation becomes

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) + B\left[y(t) - y(t-p)\right]. \tag{1}$$

Stability of periodic orbit

- The compact **monodromy operator**

$$S_p : C\left([-p, 0], \mathbb{R}^N\right) \to C\left([-p, 0], \mathbb{R}^N\right)$$

  captures how the system (1) evolves under a timestep p.
- The non-zero eigenvalues of S_p determine the stability of x_* and are called the **Floquet multipliers**.
- Count **geometric multiplicity** of Floquet multiplier μ by counting solutions of (1) of the form $y(t+p) = \mu y(t)$.

Invariance principle [Schneider & dW '21]

Geometric multiplicity of Floquet multiplier 1 is preserved under Pyragas control.

Proof: $y(t+p) = y(t)$ is a solution of

$$\dot{y}(t) = \partial_1 f(x_*(t), t)y(t) + B\left[y(t) - y(t-p)\right]$$

if and only if it is a solution of

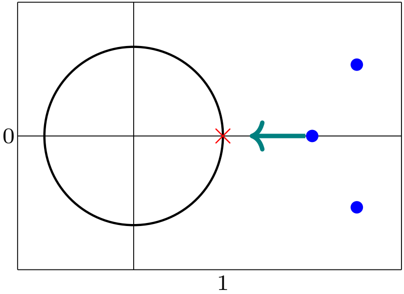
$$\dot{y}(t) = \partial_1 f(x_*(t), t)y(t).$$

Odd number limitation (cf. [Nakajima '97])

If uncontrolled system has no Floquet multiplier 1 and odd number of Floquet multipliers larger than 1: Pyragas control fails to stabilize.

Introduce homotopy parameter $\alpha \in [0, 1]$:

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) + \alpha B \left[y(t) - y(t-p) \right].$$



Odd number limitation does not apply to autonomous systems!

Always trivial multiplier 1 since $\dot{x}_*$ is always a solution of

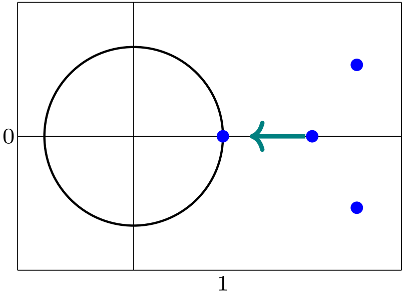
$$\dot{y}(t) = f'(x_*(t))y(t).$$

Example: stabilization close to Hopf bifurcation [Fiedler et al. '07]

But: need a non-scalar control gain: if uncontrolled system has any Floquet multiplier larger than 1, x_* is an unstable solution of

$$\dot{x}(t) = f(x(t)) + b\,[x(t) - x(t-p)]$$

with $b \in \mathbb{R}$ [Schneider & dW '21]



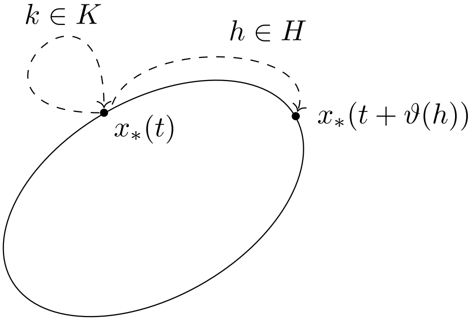
1. Delay differential equations as dynamical systems
2. Limitations to Pyragas control
3. Control of spatio-temporal patterns

Assume that the autonomous ODE

$$\dot{x}(t) = f(x(t))$$

is equivariant with respect to a compact group $\Gamma \subseteq GL(N, \mathbb{R})$, i.e.

$$f(\gamma x) = \gamma f(x) \qquad \text{for all } x \in \mathbb{R}^N, \ \gamma \in \Gamma.$$



Given a periodic solution x_* with *minimal* period $p > 0$:

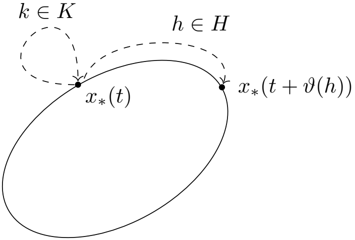
- group $K \subseteq \Gamma$ of spatial symmetries
- group $H \subseteq \Gamma$ of spatio-temporal symmetries
- map $\vartheta : H \to \mathbb{R}/(\mathbb{Z}p)$

Equivariant control term

$\dot{x}(t) = f(x(t)) + B\left[x(t) - hx(t - \vartheta(h))\right]$

with $B \in \mathbb{R}^{N \times N}$.

Pattern selective in networks of coupled oscillators [Schneider, '13].



The solution x_* is a **discrete wave** if $H/K \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$.

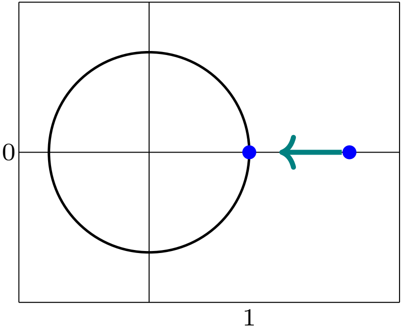
Given a $h \in H$, define **twisted monodromy operator** as

$$h^{-1} Y_{\vartheta(h)}$$

where Y_t is the fundamental solution of $\dot{y}(t) = f'(x_*(t))y(t)$.

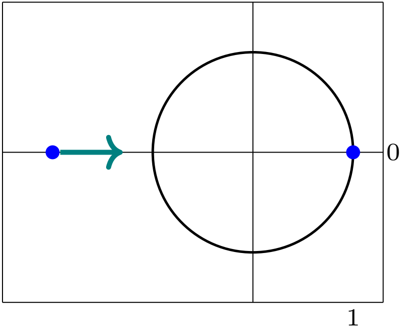
Stabilization of discrete waves: idea

Without symmetry: monodromy operator Y_p



Pyragas control with scalar control gain fails to stabilize.

With symmetry: twisted monodromy operator $h^{-1} Y_\vartheta(h)$



Equivariant control with scalar control gain can stabilize.

Stabilization of discete waves [dW, '21]

Assume that x_* is a discrete wave and that there exists a spatio-temporal symmetry $h \in H$ such that the twisted monodromy operator $h^{-1} Y_{\vartheta(h)}$ satisfies:

- the eigenvalue $1 \in \sigma(h^{-1} Y_{\vartheta(h)})$ is algebraically simple and $h^{-1} Y_{\vartheta(h)}$ has no other eigenvalues on the unit circle;
- if $\mu \in \sigma(h^{-1} Y_{\vartheta(h)})$ and $|\mu| > 1$, then $-e^2 < \mu < -1$.

Then x_* is a stable solution of

$$\dot{x}(t) = f(x(t)) + b\left[x(t) - hx(t - \vartheta(h))\right]$$

for some $b < 0$.

Characteristic matrix function

Definition [Kaashoek & Verduyn Lunel]

Let $T : X \to X$ a bounded linear operator on a complex Banach space X and let $\Delta : \mathbb{C} \to \mathbb{C}^{N \times N}$ an analytic matrix-valued function. Then Δ is a **characteristic matrix function** for T if there exist analytic functions

$$E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus X)$$

whose values are bijective operators, and such that

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix} = E(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I - zT \end{pmatrix} F(z)$$

for all $z \in \mathbb{C}$.

Δ 'summarizes' the spectral information of the operator T:

- Determine existence and algebraic multiplicity of eigenvalues from

$$\det \Delta(z) = 0.$$

- Determine geometric multiplicity of eigenvalues from $\dim \ker \Delta(z)$.

Theorem [Kaashoek & Verduyn Lunel, '21]

Assume that bounded operator $T : X \to X$ is of the form

$$T = V + R$$

with $V : X \to X$ a Volterra operator and $R : X \to X$ an operator of finite rank N. Then there exists a characteristic matrix function

$$\Delta : \mathbb{C} \to \mathbb{C}^{N \times N}$$

for T.

Applications to feedback control

Without symmetry [Kaashoek & VL '94 & '21]

The monodromy operator S_p of the linearized equation

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - y(t-p)\right]$$

has a characteristic matrix.

With symmetry [dW '21]

The twisted monodromy operator $h^{-1}S_{\vartheta(h)}$ of the linearized equation

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - hy(t-\vartheta(h)p)\right]$$

has a characteristic matrix function.

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